Mathematical Reviews

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HISTORY

Wygodski, M. The mathematics of the ancient Babylonians. Uspekhi Matem. Nauk 8, 293–335 (1941). (Russian) [MF 5081]

Neugebauer, O. On a special use of the sign "zero" in cuneiform astronomical texts. J. Amer. Oriental Soc. 61, 213–215 (1941). [MF 6074]

It is shown that the symbol in question is used not only as "zero" but also to separate tens from units in different places. *O. Schmidt* (Providence, R. I.).

Frajese, Attilio. I passi matematici di Platone. Boll. Un. Mat. Ital. (2) 3, 62–70 (1940). [MF 5576]

Frajese, Attilio. Introduzione allo studio degli Elementi di Euclide. Period. Mat. (4) 20, 137–154 (1940). [MF 5372]

Conte, Luigi. Il cosidetto XIV Libro degli Elementi. Period. Mat. (4) 21, 113–127 (1941). [MF 5384]

Italian translation, with comments, of Euclid's Elements, book XIV. *O. Neugebauer* (Providence, R. I.).

Vacca, Giovanni. Sugli specchi ustori di Archimede. Boll. Un. Mat. Ital. (2) 3, 71–73 (1940). [MF 5577]

Conte, Luigi. Dalla "Sezione del cono" di Sereno. IV. Period. Mat. (4) 20, 218–239 (1940). [MF 5375]

Concluding communication. Part III appeared in the same volume, pp. 1–23; cf. these Rev. 1, 289.

Diller, Aubrey. The parallels on the Ptolemaic maps. Isis 33, 4–7 (1941). [MF 5105]

McCoy, John Calvin. Manuel Moschopoulos's treatise on magic squares. Scripta Math. 8, 15–26 (1941). [MF 5098]

Translation of the French translation of the Greek text [Tannery, Mémoires scientifiques, vol. 4, pp. 25–60]. *O. Neugebauer* (Providence, R. I.).

Cassina, Ugo. Sulle equazioni cubiche di Al Biruni. Period. Mat. (4) 21, 3–20 (1941). [MF 5379]

Cassina, Ugo. La trisezione dell'angolo in Al Biruni. Period. Mat. (4) 21, 77–87 (1941). [MF 5381]

Italian translation of the relevant parts of C. Schoy, Die trigonometrischen Lehren des . . . Al-Biruni [Hannover, 1927], with critical comments. *O. Neugebauer.*

Babini, José. On the linear systems of Leonardo Pisano. Archeion 23, 57–61 (1941). [MF 5086]

Remarks concerning K. Vogel's paper "Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten [Deutsche Math. 5, 217–240 (1940); cf. these Rev. 2, 114].

Santaló, Luis A. Nicolas Tartaglia and the resolution of the equation of third order. Math. Notae 1, 26–33 (1941). (Spanish) [MF 5009]

Candido, Giacomo. Le risoluzioni della equazione di quarto grado. Period. Mat. (4) 21, 21–44 (1941). [MF 5380]

Candido, Giacomo. Le risoluzioni della equazione di quarto grado (Ferrari-Eulero-Lagrange). Period. Mat. (4) 21, 88–106 (1941). [MF 5382]

Candido, Giacomo. Le risoluzioni della equazione di quarto grado (Fagnano). Period. Mat. (4) 21, 151–176 (1941). [MF 5386]

Peters, Theodor. Über Näherungskonstruktionen und Mechanismen im ersten Buch der Harmonik Keplers und seine Forderung nach Beschränkung der Konstruktionsmittel allein auf Zirkel und Lineal. Deutsche Math. 6, 118–132 (1941). [MF 5813]

The author collects various speculations of Kepler about the possibility of measuring the edge of the regular heptagon, etc., by the radius of the circumscribed circle. Some earlier approximate solutions are discussed. I do not see the justification for the author's conclusion that Kepler had a clear understanding of the limitation of the problem. *O. Neugebauer* (Providence, R. I.).

Brun, Viggo. The quadrature of the circle. Norsk Mat. Tidsskr. 23, 41–53 (1941). (Norwegian) [MF 4984]

Agostini, Amedeo. Rileggendo la "Geometria speciosa" di Pietro Mengoli. Period. Mat. (4) 20, 313–327 (1940). [MF 5378]

Report about discoveries in the methods of the calculus by Mengoli (Bologna, 1628–1686). *O. Neugebauer.*

Agostini, A. Le serie sommate da Pietro Mengoli. Boll. Un. Mat. Ital. (2) 3, 231–251 (1941). [MF 5594]

Hofmann, Josef Ehrenfried. On the discovery of the logarithmic series and its development in England up to Cotes. Nat. Math. Mag. 14, 37–45 (1939). [MF 5250]

Procissi, Angiolo. Lettere di Giovanni Ceva ad A. Magliabechi con note bibliografiche. Period. Mat. (4) 20, 289–312 (1940). [MF 5377]

Bibliography of unpublished letters of Ceva. To one of them is attached a short treatise on the principles of mechanics, which is here translated into Italian. *O. Neugebauer* (Providence, R. I.).

Gewirtz, Alan. Experience and the non-mathematical in the Cartesian method. J. Hist. Ideas 2, 183–210 (1941). [MF 5196]

Sergescu, P. Quelques dates remarquables dans l'évolution des mathématiques en France. Monografii Mat., fasc. 4, 32 pp. (1941). [MF 5554]

Krakeur, Lester Gilbert and Krueger, Raymond Leslie. The mathematical writings of Diderot. Isis 33, 219–232 (1941). [MF 5106]

Loria, Gino. Gli "Acta Eruditorum" durante gli anni 1682–1740 e la storia delle matematiche. Archeion 23, 1–35 (1941). [MF 5085]

***Keyser, Cassius Jackson. Charles Sanders Peirce as a pioneer.** Galois Lectures, Scripta Mathematica Library, no. 5, pp. 87–112. New York, 1941.

McColley, Grant. George Valla: an unnoted advocate of the geo-heliocentric theory. Isis 33, 312–314 (1941). [MF 5107]

***Ludendorff, H. Die astronomischen Inschriften der Maya.** Astronomical Papers dedicated to Elis Strömgren (Festschrift für Elis Strömgren), pp. 143–162. Einar Munksgaard, Copenhagen, 1940.

Kuzmin, R. O. The mathematical works of S. N. Bernstein. Uspekhi Matem. Nauk 8, 3–7 (1941). (Russian) [MF 5266]

Smirnoff, V. I. and Soboleff, S. L. Obituary: Nikolai Maksimovič Gunther. 1871–1941. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 193–197; list of publications, 197–202 (1941). (Russian) [MF 5169]

Aitken, A. C. Obituary: Dr. E. L. Ince. Nature 148, 309–310 (1941). [MF 5327]

Whittaker, E. T. Obituary: Edward Lindsay Ince. 1891–1941. J. London Math. Soc. 16, 139–143; list of publications, 143–144 (1941). [MF 5136]

Young, A. W. Obituary: Edward Lindsay Ince, M.A., D.Sc., F.R.S.E. Proc. Edinburgh Math. Soc. (2) 6, 263–264 (1941). [MF 5867]

Comrie, P. Obituary: George Lawson, M.A. Proc. Edinburgh Math. Soc. (2) 6, 261–262 (1941). [MF 5866]

Milne, E. A. Obituary: Augustus Edward Hough Love. 1863–1940. J. London Math. Soc. 16, 69–78; list of publications, 78–80 (1941). [MF 5905]

Milne, E. A. Obituary: Augustus Edward Hough Love. 1863–1940. Obit. Notices Roy. Soc. London 3, 467–479; list of publications, 480–482 (1941). (1 plate) [MF 6175]

Dougall, John. Obituary: Robert Franklin Muirhead, B.A., D.Sc. Proc. Edinburgh Math. Soc. (2) 6, 259–260 (1941). [MF 5865]

Chaundy, T. Obituary: E. G. C. Poole. J. London Math. Soc. 16, 125–130; list of publications, 130 (1941). [MF 5134]

v. Koppenfels, Werner. Obituary: Georg Prange. Jber. Deutsch. Math. Verein. 51, 1–13; list of publications, 13–14 (1941).

Hofmann, Jos. E. Johannes Tropfke (14.X.1866 bis 10.XI. 1939). Deutsche Math. 6, 114–117; list of publications, 117–118 (1941). (1 plate) [MF 5812]

Allen, E. S. The scientific work of Vito Volterra. Amer. Math. Monthly 48, 516–519 (1941). [MF 5713]

Picard, Émile. Obituary: Vito Volterra. C. R. Acad. Sci. Paris 211, 309–312 (1940). [MF 5350]

Whittaker, E. T. Obituary: Vito Volterra. 1860–1940. J. London Math. Soc. 16, 131–139 (1941). [MF 5135]

Turnbull, H. W. Obituary: Alfred Young. 1873–1940. J. London Math. Soc. 16, 194–207; list of publications, 207–208 (1941). [MF 6019]

ALGEBRA

Linear Algebra

Barba, G. Una dimostrazione elementare del teorema di Laplace per lo sviluppo di un determinante. Boll. Un. Mat. Ital. (2) 3, 255–256 (1941). [MF 5595]

Milne-Thomson, L. M. Determinant expansions. Math. Gaz. 25, 130–135 (1941). [MF 5730]

This paper of an expository character gives a proof of L. H. Rice's expansion theorem on determinants [Amer. J. Math. 42, 237 (1920)]. The author shows that Rice's theorem contains a simultaneous justification for all special expansions of a determinant and accordingly is deserving of more attention by writers on the subject of determinants. *J. Williamson* (Flushing, N. Y.).

Chakrabarti, Satish Chandra. On a few recurrents. J. Indian Math. Soc. (N.S.) 5, 18–26 (1941). [MF 5109]

The author evaluates two determinants whose elements below the main diagonal are zero, while those above are certain elementary symmetric functions of $1, a, a^2, a^3, \cdots, a^{n-1}$. The value of a third determinant, whose elements are rational functions of a (which are difficult to describe), is deduced from the preceding results. *D. H. Lehmer.*

Castoldi, Luigi. Alcune osservazioni sui numeri complessi e sui quaternioni. Period. Mat. (4) 20, 178–185 (1940). [MF 5373]

Conditions that matrices T_a in a Euclidean space of n dimensions satisfy the conditions (summation on h) (a) the product $T_a T'_{ah}=0$ only when at least one of the factors is zero; (b) the product $T_a T'_{ah}$ has the same structure as T and T'; (c) the product $T_a T'_{ah}$ is commutative. It is shown that only for $n=2$ all conditions can be satisfied, and that by the matrix

$$\begin{Vmatrix} a & b \\ -b & a \end{Vmatrix},$$

which corresponds to complex numbers. For $n=4$ the con-

ditions (a) and (b) lead to

$$\begin{Vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{Vmatrix},$$

which leads to quaternions. *D. J. Struik.*

Kurosaki, Tiyoko. Über die mit einer Kollineation vertauschbaren Kollineationen. Proc. Imp. Acad. Tokyo 17, 24–28 (1941). [MF 4335]

Let A be a square matrix with coefficients in a given field K. The author studies the matrices P with coefficients in K which satisfy an equation $PA=cAP$, where c is a scalar. For example, the number of linearly independent solutions P of this equation for fixed c is determined.

R. Brauer (Toronto, Ont.).

Rados, Gusztáv. Vertauschbare Hermitesche Matrizen und die zugehörigen Hermiteschen Formen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 1–6 (1940). (Hungarian. German summary) [MF 4443]

Summary of some results concerning the "Faltung" of two permutable Hermitian matrices. *O. Szász.*

v. Dávid, Lajos. Verallgemeinerung iterativer Matrizen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 59, 62–71 (1940). (Hungarian. German summary) [MF 4446]

A matrix $(a_{\mu\nu})$, $\mu=1, 2, \cdots, n$; $\nu=0, 1, 2, \cdots$, of n rows and infinitely many columns is called iterative if the absolute maximal (minimal) element of the νth column is a preassigned function of the $(\nu-1)$st column. Subjecting this function to certain assumptions, the author proves that as $\nu\to\infty$ the limits $\lim_{\nu\to\infty} a_{\mu\nu}$, $\mu=1, 2, \cdots, n$, exist and are equal. *O. Szász* (Cincinnati, Ohio).

Suschkewitsch, A. Groupes généralisés des matrices singulières. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 3–11 (1940). (Russian. French summary) [MF 4728]

Every square matrix A of order m and rank $r<m$ can be represented as KL, where K is an m-r matrix (m rows and r columns) and L is an r-m matrix, both being of rank r. K can be taken as AX and L as YA, where X (or Y) may be chosen arbitrarily subject only to the condition that K (or L) be of rank r. If $XL=I_d$ and $KY=I_s$, it is shown that both these matrices are idempotent and that $AI_d=I_sA=A$. Also $I'=LX=YAX=YK$ is idempotent. If $I=PQ$ is an arbitrary idempotent matrix then it is a right identity for all matrices $A=XQ$, X being subject to the above condition; the totality $\mathfrak{A}_Q$ of such matrices A contains a subset $\mathfrak{G}_{P,Q}$ for which I is also a left identity and $\mathfrak{G}_{P,Q}$ is an ordinary group. The complex $\mathfrak{R}_r=\sum_Q\mathfrak{A}_Q$ does not form a group, but $\mathfrak{M}_{m,r}=\mathfrak{R}_r+\mathfrak{R}_{r-1}+\cdots+\mathfrak{R}_0$ $(0\leqq r\leqq m-1)$ does, as follows from the fact that every matrix of order m and rank $r-1$ can be expressed as the product of two matrices of order m and rank r $(r<m)$. *M. S. Knebelman* (Pullman, Wash.).

Suschkewitsch, A. Groupes généralisés de quelques types des matrices infinies. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 115–120 (1940). (Ukrainian. French summary) [MF 4738]

The paper deals with real infinite matrices $A=(a_{\alpha\beta})$, the row vectors being orthonormal in Hilbert space, that is, $\sum_1^\infty a_{\alpha\lambda}a_{\beta\lambda}=\delta_{\alpha\beta}$. If the system of row vectors is complete, A is orthogonal; otherwise the system is incomplete and A is said to be left semiorthogonal; A is proper if it takes an infinite number of vectors to complete the system. The main theorem proved is that all proper left semiorthogonal matrices form a generalized group with a unique right inverse and infinitely valued left inverse. There is also a brief treatment of proper left semiorthogonal matrices in which (1) every row vector has one component equal to 1 and the rest to zero, (2) every column has at most one 1 in it, (3) there is an infinite number of zero columns. Such a matrix defines a deficient substitution on a denumerably infinite set of elements [cf. the following review]. *M. S. Knebelman.*

Suschkewitsch, A. Untersuchungen über unendliche Substitutionen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 18, 27–37 (1940). (Russian. German summary) [MF 4759]

The substitutions on a denumerably infinite set of elements are divided into four classes: ordinary, if the lower line contains each element just once; deficient, if it contains only a proper subset of elements without repetitions. If the missing elements form an infinite set the substitution is absolutely deficient; it is excessive (überschüssig) if the lower line contains all elements, some of them repeated. If each element occurs an infinite number of times, it is absolutely excessive. Mixed substitutions are those in which the lower line contains a proper subset of elements but some are repeated. All ordinary substitutions form an ordinary group. Every ordinary substitution A can be represented as KL, where K is deficient and L is excessive. All deficient substitutions form a left semigroup and all excessive ones a right semigroup. If K is deficient and L excessive, KL is, in general, mixed and conditions are derived on K and L so that KL shall be of prescribed class. Among some other theorems it is proved that, if G is an ordinary group and its identity is represented by KL as above, then LGK is also an ordinary group of mixed substitutions simply isomorphic with G. *M. S. Knebelman* (Pullman, Wash.).

Gorschkow, D. S. Kubische Körper und symmetrische Matrizen. C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 842–844 (1941). [MF 5513]

If a cubic field (over the field of rational numbers) is generated by a root of the characteristic equation of a symmetric matrix (with rational elements), it is easily seen that every number in the field is a root of such an equation. The author proves that a necessary and sufficient condition for this state of affairs is that the field contain three integers α, β, γ such that (1) $N(\alpha)/\alpha+N(\beta)/\beta+N(\gamma)/\gamma=0$. Here $N(\theta)$ stands for the norm of θ. It is easy to identify any cubic equation as the characteristic equation of a symmetric matrix, but some of the elements of this matrix may be irrational. Condition (1) is proved to be sufficient by showing that the elements of the matrix constructed for β/α are all rational when (1) is satisfied. The necessity of (1) is proved similarly. *H. W. Brinkmann* (Swarthmore, Pa.).

Abstract Algebra

***Birkhoff, Garrett and MacLane, Saunders. A Survey of Modern Algebra.** Macmillan Company, New York, 1941. xi+450 pp. $3.75.

This is a text on modern algebra that is particularly suited for a first year graduate course or for an advanced undergraduate course. A very striking feature of the book is its broad point of view. There are contacts with many branches of mathematics and so it can serve as an intro-

duction to nearly the whole of modern mathematics. Thus there is a careful development of real numbers, such as Dedekind cuts, and such set-theoretic concepts as order, countability and cardinal number are discussed. Throughout the study of matrices and quadratic forms the geometric point of view is emphasized. There is also contact with the field of mathematical logic in the chapter on the algebra of classes and with the ideas of topology in the proof of the fundamental theorem of algebra.

The following is the table of contents by chapters: I. The integers. II. Rational numbers and fields. III. Real numbers. IV. Polynomials. V. Complex numbers. VI. Group theory. VII. Vectors and vector spaces. VIII. The algebra of matrices. IX. Linear groups. X. Rank and determinants. XI. Algebra of classes. XII. Transfinite arithmetic. XIII. Rings and ideals. XIV. Algebraic number fields. XV. Galois theory. As can be judged from this outline, concrete systems with which the student is familiar are studied first and then their properties are formulated as axioms leading thereby to the definition of important abstract algebraic systems. This has been done with great skill in the text and moreover the large number of excellent exercises should enable the student to keep a firm hold on the theory. There is a wide gap between the first chapter dealing with elementary properties of integers and the last chapter proving the insolvability of quintic equations by radicals. One is confident, however, that a student who has made the journey over the intermediate chapters will have achieved the maturity needed to follow the intricacies of the Galois theory.

The most serious criticism that the reviewer has to make is a rather minor one: The theory of determinants is developed only for matrices with entries in a field. Because of the applications we believe it would have been worthwhile to treat the general case in which the entries belong to any commutative ring. The book is very clearly written and seems to be free from error. *N. Jacobson.*

Dilworth, R. P. The arithmetical theory of Birkhoff lattices. Duke Math. J. **8**, 286–299 (1941). [MF 4593]

The author wishes to know when the number of irreducible components occurring in a representation $a=x_1 \cap \cdots \cap x_n$ of an arbitrary element a of a lattice L, as the irredundant meet of meet-irreducible elements, should be an invariant $n(a)$ of a. It is known (Theorem of Kurosch-Ore) that this is true if L is modular; this result contains theorems of E. Noether on ideals and R. Remak on groups. The author generalizes this result and obtains a partial converse, as follows. He shows that, if L is upper semi-modular (a "Birkhoff lattice") and suitable chain conditions hold, then the above uniqueness theorem is valid if and only if the sublattice generated by the elements covering any fixed b is a modular interval sublattice (that is, convex sublattice) of L.
G. Birkhoff (Cambridge, Mass.).

Frink, Orrin, Jr. Representations of Boolean algebras. Bull. Amer. Math. Soc. **47**, 755–756 (1941). [MF 5481]

A remarkably brief and elegant proof is given of the well-known theorem that every Boolean algebra is isomorphic with an algebra of sets. The proof is based on a simple new set of four postulates on 0, $x \cap y$ and x'.
G. Birkhoff (Cambridge, Mass.).

Dyer-Bennet, John. A note on finite regular rings. Bull. Amer. Math. Soc. **47**, 784–787 (1941). [MF 5489]

The author gives a complete classification of finite regular rings (in the sense of von Neumann). A finite regular ring can be expressed uniquely as the direct sum of full matrix algebras over finite fields. Using this result, a combinatory formula is given for computing the exact number of non-isomorphic regular rings of any given finite order.
G. Birkhoff (Cambridge, Mass.).

Pate, Robert S. Rings with multiple-valued operations. Duke Math. J. **8**, 506–517 (1941). [MF 5186]

This investigation has for its object rings for which the postulates are relaxed to allow the sums $a+b$ and the products ab to be sets of elements instead of single elements. Some special ideals are shown to form a lattice. These, called Q-ideals, are defined in terms of a fixed element q, and contain with the element r all solutions x of $q+x \supset r$ and $r+x \supset q$. A basis for these ideals is found. They are shown to satisfy the Dedekind identity: $A \cup (B \cap C)=(A \cup B) \cap C$ if $C \supset A$. The adjunction of an indeterminate x is studied, and the surprising fact uncovered that the extended domain is an integral domain in this new (weaker) sense only if it is an integral domain in the classical sense.
H. H. Campaigne (Minneapolis, Minn.).

Zorn, Max. Alternative rings and related questions I: existence of the radical. Ann. of Math. (2) **42**, 676–686 (1941). [MF 4967]

An alternative ring is one in which the function $(a, b, c)=(ab)c-a(bc)$ is an alternating function of its arguments. The structure of semi-simple systems of this type has been studied in an earlier paper by Zorn [Abh. Math. Sem. Hansischen Univ. **8**, 123–147 (1932)]. In the present paper the author considers the theory of the radical. Let $(a\backslash 0)$ denote the set of solutions of $ax=0$. An alternative ring S is called hypercomplex if it satisfies the following conditions: (1) every chain $a^i S$ is finite; (2) every ascending chain $(a^i\backslash 0)$ is finite; (3) every descending chain $(a_i\backslash 0)$ is finite. For such a ring there exists a maximal set of orthogonal idempotent elements e_i $(i=1, \cdots, n)$ and a decomposition of S as $\sum S_{ij}$, $i, j=0, \cdots, n$, where S_{ik} is the set of elements x_{ik} such that $e_i x_{ik}=\delta_{ij} x_{ik}$ and $x_{ik} e_j=\delta_{kj} x_{ik}$. Let N be the set of elements $\sum n_{ik}$ such that $n_{ik} S_{ki}$ consists exclusively of nilpotent elements. Then the principal result is that N is an ideal and coincides with the set of properly nilpotent elements. Thus the set of properly nilpotent elements is an ideal, "the radical of S." *N. Jacobson.*

Witt, Ernst. Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe. Abh. Math. Sem. Hansischen Univ. **14**, 289–322 (1941). [MF 5448]

This paper deals with Cartan's classification of all semi-simple Lie algebras L over the field of all complex numbers. By the results of Cartan and Weyl it suffices to consider simple Lie algebras L given as special distributive systems by a basis H, e_a with the relations $\mathfrak{H} \circ \mathfrak{H}=0$, $\mathfrak{h} \circ e_a=\mathfrak{h}\mathfrak{a} \circ e_a$, $e_{-a} \circ e_a=\mathfrak{a}$. The product $\mathfrak{h}\mathfrak{a}$ means the ordinary inner product of vectors; the $\mathfrak{a}$ are the characteristic roots of the ring L. Weyl's analysis of Lie algebras L shows that the finite set of N real vectors $\mathfrak{a}$ determines a vector diagram $\mathfrak{B}$ in a real space. This vector diagram contains with $\mathfrak{a}$ only $-\mathfrak{a}$; moreover $\mathfrak{B}$ is carried into itself by reflections on the origin through the hyperplanes perpendicular to each $\mathfrak{a}$. The vector diagrams $\mathfrak{B}$ determined by Lie algebra are characterized, among others, by the special property that (*) $2\mathfrak{a}\mathfrak{b}(\mathfrak{a}^2)^{-1}$ is integral for each $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{B}$. Actually L determines a class of similar diagrams $\mathfrak{B}$. The author presents an elegant proof of a result of van der Waerden stating that each properly restricted class of diagrams $\mathfrak{B}$ determines a class of isomor-

phic Lie algebras. The actual construction of all diagrams satisfying (*) is given by induction. Suppose that $\mathfrak{R}$ is the linear space determined by a vector diagram $\mathfrak{B}$. The hyperplanes $\mathfrak{A}_i$ perpendicular to the $\mathfrak{a}_i$ dissect the space $\mathfrak{R}$ into at most 2^N convex regions $\mathfrak{B}$. Let n be the number of faces of a typical convex $\mathfrak{B}$. Then the explicit knowledge of all diagrams $\mathfrak{B}$ with $n \leqq 4$ suffices to construct all simple Lie algebras L. The classification of the set of all similar diagrams uses as a tool the group $\mathfrak{G}$ of reflections determined by $\mathfrak{B}$ and its associated quadratic form $f = -\sum \xi_i \xi_k \cos \pi/m_{ik}$, where the π/m_{ik} are the angles between the ith and jth faces of $\mathfrak{B}$; m_{ik} are positive integers, $m_{ik} = \infty$ is admitted and occurs for unbounded convexes. The author's discussion of the group $\mathfrak{G}$ is closely related to the work of H. S. M. Coxeter [e.g., Ann. of Math. (2) 35, 588–621 (1934)].

O. F. G. Schilling (Chicago, Ill.).

Chang, Ho-Jui. Über Wittsche Lie-Ringe. Abh. Math. Sem. Hansischen Univ. 14, 151–184 (1941). [MF 5442]

Let k be a field of characteristic $p > 2$. Then the quantities e_i, $i = 0, \cdots, p-1$, with the relations $e_i \circ e_j = (j-1)e_{i+j}$, $i+j$ mod p, determine a Lie algebra L over k. This algebra is isomorphic to the set of polynomials $f, g, \cdots$, in $k[x]/(x^p)$, where $f \circ g = fg' - gf'$. Using this representation of L the author shows that each mapping $f(x) \to f[\phi(x)]/\phi'(x)$, $\phi(x) = \sum_{i=1}^{p-1} a_i x^i$, $a_1 \neq 0$, determines an automorphism of L. In case that $p \geqq 5$ the group of all automorphisms of L is isomorphic to the group of all polynomials ϕ, ψ, where $\phi \times \psi = \phi(\psi)$. The author then develops a detailed theory of the irreducible representations of L in case that $p \geqq 3$ and k is algebraically complete. The representations are shown to depend on p parameters (characteristic roots) and their degrees are bounded by $p^{(p-1)/2}$. The methods used are similar to the ones developed by Zassenhaus [Abh. Math. Sem. Hansischen Univ. 12, 1–100 (1939); J. Reine Angew. Math. 182, 150–155 (1940); cf. these Rev. 2, 121].

O. F. G. Schilling (Chicago, Ill.).

Dieudonné, Jean. Sur les systèmes hypercomplexes. C. R. Acad. Sci. Paris 211, 172–174 (1940). [MF 5342]

A study of the structure of rings and minimal and maximal conditions for ideals in the case where there is a radical. Instead of the classical decomposition into a semi-simple ring and the radical, the author introduces a method of step-wise decomposition in two-sided ideals, the advantage of which is to give some information on the multiplication table of the given ring. *C. Chevalley* (Princeton, N. J.).

Moriya, Mikao und Kobayasi, Yosi. Eine notwendige Bedingung für die eindeutige Primfaktorzerlegung der Ideale in einem kommutativen Ring. Proc. Imp. Acad. Tokyo 17, 129–133 (1941). [MF 5393]

Let $\mathfrak{O}$ be a commutative ring containing non-trivial ideals. Suppose that each ideal is a unique power product of a finite number of prime ideals $\mathfrak{p}$. The authors prove that the ring $\mathfrak{O}$ has the following properties: (1) it contains an identity element for multiplication; (2) the weak chain conditions hold for ideals; (3) there is no proper ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$; and (4) the radical of $\mathfrak{O}$ is a prime ideal.

O. F. G. Schilling (Chicago, Ill.).

Kobayasi, Yosi und Moriya, Mikao. Eine hinreichende Bedingung für die eindeutige Primfaktorzerlegung der Ideale in einem kommutativen Ring. Proc. Imp. Acad. Tokyo 17, 134–138 (1941). [MF 5394]

The authors prove that the properties (1)–(4) of the above review imply in turn the unique factorization for ideals. The methods applied are the familiar ones of the additive ideal theory. Examples are given to prove the independence of properties (1)–(4). *O. F. G. Schilling* (Chicago, Ill.).

Robinsohn, Abraham. On a certain variation of the distributive law for a commutative algebraic field. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 93–101 (1941). [MF 5630]

The author defines as a "quasi-field" a set having the usual properties of a commutative field, except that the distributive law is replaced by the relation $a(b_1+b_2+\cdots+b_n) = ab_1+\cdots+ab_n$ for some fixed $n(>2)$. He studies such quasi-fields and, in particular, introduces ideals in them. He further shows that it is possible, in any quasi-field, to define a multiplication so that the quasi-field becomes an ordinary ring and such that the ideals in the quasi-field become ideals in this ring. A similar definition is made for "quasi-rings" and "quasi-domains-of-integrity," and it is shown that one of the latter can always be imbedded in a quasi-field in the way that an ordinary domain of integrity can be imbedded in an ordinary field.

H. W. Brinkmann (Swarthmore, Pa.).

Delone, B. N. On the geometry of Galois theory. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 52–62. (Russian) [MF 3502]

The author gives a compact exposition of a geometric approach to the study of the Galois theory. He considers an n-dimensional complex space K_n and an n-dimensional point lattice $[\omega_1, \cdots, \omega_n]$ in it, that is, the totality of points $\omega = u_1\omega_1 + \cdots + u_n\omega_n$, where u_i are integers and $\omega_i = (\omega_i', \cdots, \omega_i^{(n)})$ are n points the determinant of whose coordinates is different from zero. He defines the sum, the difference, or the product of two points as the point whose coordinates are the sum, the difference, or the product of the corresponding coordinates of the two points. A point lattice in K_n is said to repeat under multiplication if the product of any two points of the lattice is again a point of the lattice. Coordinates of points in such a lattice are integral algebraic numbers. A lattice formed from a given lattice $[\omega_1, \cdots, \omega_n]$ by incorporating additional points $\omega = \mu_1\omega_1 + \cdots + \mu_n\omega_n$, where μ_i are rational fractions, is said to be formed through "condensation." A point lattice is called maximal if it repeats under multiplication and if it possesses no condensation which repeats under multiplication. A point in K_n other than the origin is called a divisor of zero if at least one of its coordinates is zero. A point lattice containing no divisors of zero is said to be irreducible. The study [§§ 1–3] of such lattices yields a geometric counterpart of some familiar relations among the algebraic numbers.

Next, by a permutation of the axes in a lattice is meant a rotation about the origin, in K_n, which leaves invariant the totality of the positive semi-axes. If the point lattice obtained by a permutation of the axes coincides with the original point lattice, it is said that this permutation carries the point lattice into itself. There can be no more than n permutations of the axes of an irreducible n-dimensional lattice, which carry it into itself. An irreducible maximal lattice of n dimensions with exactly n axes permutations which carry it into itself is called normal. Every irreducible maximal n-dimensional lattice is [Th. IV] either a normal lattice or a maximal sublattice of a normal lattice. The group G of all permutations of the axes of a normal lattice B which carry B into itself is called the Galois group of B. Finally (the correspondence theorem), to every subgroup H

of index ν of the group G there corresponds a ν-dimensional maximal sublattice of B and conversely.

The author states that his geometric method yielded a very neat solution of the "inverse problem" of the Galois theory for the five groups admitted by cubic equations and for the eleven groups admitted by the equations of fourth degree. For the details of this, one is referred to the monograph on cubic irrationalities by the author and Faddeev [cf. these Rev. 2, 349], and for further applications of the theory, to forthcoming papers.

One would like to mention here a development of the Galois theory due to Artin, which treats a Galois group simply as a finite group of automorphisms [cf. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Macmillan, New York, 1941, chap. 15; cf. these Rev. 3, 99. For a more recent and a more general version, see the notes on Artin's lectures on the Galois theory at Notre Dame University, which are being prepared by Milgram].

A. E. Ross (St. Louis, Mo.).

Schwarz, Štefan. Über einen Satz von S. Lubelski. Časopis Pěst. Mat. Fys. **69**, 146–147 (1940). (German. Czech summary) [MF 5206]

This note contains a simple proof of the following theorem of Lubelski [Acta Arith. **1**, 169–183 (1936); **2**, 250–261 (1937)]. Let $f(x)$ be an irreducible polynomial with rational integral coefficients whose root θ defines a non-normal field K' over the rational field K. Let $\phi(x)$ be the Galois resolvent of $f(x)$ and p any rational prime which does not divide the discriminant of $\phi(x)$. Then $\phi(x)$ is reducible mod p. Let K'' be the normal field defined by a root of $\varphi(x)=0$ and let n be the order of the Galois group of $K'':K$. The proof consists in noting that if $\phi(x)$ is irreducible mod p then (p) is a prime ideal of degree n in the normal field K''. The decomposition group for (p) is then cyclic, of order n, and equal to the full Galois group of $K'':K$. The Galois group then has a normal subgroup corresponding to K' which is contrary to the assumption that K' is non-normal.

D. C. Murdoch (Saskatoon, Sask.).

Krasner, Marc. Un critère de primarité. C. R. Acad. Sci. Paris **212**, 323–324 (1941). [MF 4903]

A number $\alpha=0$ of a $\mathfrak{p}$-adic number field k containing the primitive qth roots of unity, where $q=p^n$, $\mathfrak{p}\mid p$, p a rational prime, is said to be q-primary in k if $k(\alpha^{1/q})/k$ is unramified. The author's criterion for q-primarity in k gives the structure of the multiplicative group P of q-primary numbers of k completely in terms of the group H of qth powers of non-zero numbers of k, and the absolutely Abelian local subfield $k_{n,s}$ of k, which is the composite of the local field k_n of qth roots of unity and the unramified local field $k^{(s)}$ of absolute degree p^s, where $p^s\|f$ and f is the absolute degree of $\mathfrak{p}$.

R. Hull (Vancouver, B. C.).

MacLane, Saunders and Schilling, O. F. G. Normal algebraic number fields. Trans. Amer. Math. Soc. **50**, 295–384 (1941). [MF 5144]

The paper is (as the authors say in the introduction) an exploration of the potentialities of the concept of factor sets and algebras in the study of arbitrary normal fields. Let K/k be a normal extension of the basic field k (a field of algebraic numbers) and let Γ be the Galois group of K/k. An ideal algebra $\mathfrak{S}$ over k is defined when to every prime $\mathfrak{p}$ in k (finite or infinite) there is associated a class $\mathfrak{S}_\mathfrak{p}$ of normal simple algebras over $k_\mathfrak{p}$ (the $\mathfrak{p}$-adic field), in such a way that $\mathfrak{S}_\mathfrak{p}\sim k_\mathfrak{p}$ for almost all $\mathfrak{p}$. The ideal algebras form an Abelian group, which contains as a subgroup the group of classes of normal simple algebras over k. If $Kk_\mathfrak{p}$ is, for every $\mathfrak{p}$, a splitting field of $\mathfrak{S}_\mathfrak{p}$, we say that K splits $\mathfrak{S}$. The main interest of the paper centers on the computation of the index J of the group of the actual algebras split by K in the group of the ideal algebras split by K. If K/k is cyclic, J reduces to the index in the group of ideals relatively prime to the conductor of the group $\alpha N_{K/k}(\mathfrak{A})$, where $\alpha\equiv 1$ modulo the conductor and $\mathfrak{A}$ runs over the ideals of K prime to the conductor. Therefore, in this case, J is equal to the degree of K/k. In the general case, J is equal to the l.c.m. of the orders of the elements of Γ [Theorem 4.1, p. 403]. The index J may be (just as in the cyclic case) decomposed in two parts, one which refers to the factor-sets of ideals and the other one, $J(E)$, which refers to the factor-sets of units [Theorem 11.2, p. 325]. The first factor may be expressed by factors involving factor-sets of classes of ideals and certain "deficiency-indices" [p. 339]. The index $J(E)$ cannot in the general case be entirely reduced to an index computable in a Herbrand-Minkowski group of units [Theorem 19.1, p. 346]. The authors give also the definition of a generalized norm-residue symbol and of a generalized Artin symbol.

C. Chevalley (Princeton, N. J.).

Perlis, Sam. Scalar extensions of algebras with exponent equal to index. Bull. Amer. Math. Soc. **47**, 670–676 (1941). [MF 5064]

The author investigates the effect of scalar extensions upon the index and exponent of normal simple algebras. The following theorem is proved: "Let F be a field, p a prime and W any separable extension of F. Every normal simple algebra A of index p^n and exponent p^n (n variable) over every such field W has the property that the index and exponent of A_K are equal, for every field K of finite degree over W, if and only if this is true for every field K which is cyclic of degree p over W." The author illustrates by an example that the hypothesis for W/F cannot be omitted.

O. F. G. Schilling (Chicago, Ill.).

Wajnsztejn, D. Über Algebren mit endlicher Basis. Studia Math. **9**, 109–120 (1940). (German. Ukrainian summary) [MF 5260]

The principal result of this paper is a proof that every algebra, with a finite basis, over the field of complex numbers is isomorphic to a subalgebra of Clifford numbers [Amer. J. Math. **1**, 350–358 (1878)]. The main part of the proof consists in the construction of a particular basis for the algebra of all matrices of order 2^n, and the proof that this algebra is an algebra of Clifford numbers determined by a certain set of $2n$ elements of the basis. The paper is closely related to work of Newman [J. London Math. Soc. **7**, 93–99 (1932)] and Littlewood [ibid. **9**, 41–50 (1934)]. Additional references are also given by the author.

N. H. McCoy (Northampton, Mass.).

Etherington, I. M. H. Special train algebras. Quart. J. Math., Oxford Ser. **12**, 1–8 (1941). [MF 4664]

This paper gives a proof of the theorem, previously announced [Proc. Roy. Soc. Edinburgh **59**, 242–258 (1939); these Rev. **1**, 99], which states that every special train algebra is a train algebra, and illustrates the technical procedures upon an example from genetics.

M. H. Stone.

Etherington, I. M. H. Some non-associative algebras in which the multiplication of indices is commutative. J. London Math. Soc. **16**, 48–55 (1941). [MF 5903]

The author considers commutative algebras which are not in general associative but have the property that $(x^r)^s=(x^s)^r$. It is shown that the algebra with basis $a_1, a_2, \cdots, a_n$ and multiplication table $a_1^2=\lambda_1 a_2$, $a_2^2=\lambda_2 a_3$, $\cdots$, $a_{n-1}^2=\lambda_{n-1}a_n$, $a_n^2=\lambda_n a_1$, $a_\theta a_\varphi=0$ $(\theta\neq\varphi)$ has this property. An algebra is baric if it has a non-trivial homomorphic mapping on its coefficient field. A baric algebra also has the above property provided a certain invariant subalgebra (ideal) P is such that $P^2=0$. *C. C. MacDuffee* (New York, N. Y.).

Etherington, I. M. H. Duplication of linear algebras. Proc. Edinburgh Math. Soc. (2) **6**, 222–230 (1941). [MF 5863]

The linear algebras here considered, usually non-associative, have been found useful in genetics [Proc. Roy. Soc. Edinburgh. Sect. A. **59**, 242–258 (1939); cf. these Rev. **1**, 99; J. London Math. Soc. **15**, 136–149 (1940); cf. these Rev. **2**, 121; Proc. Roy. Soc. Edinburgh. Sect. B. **61**, 24–42 (1941); cf. these Rev. **2**, 237]. If the algebra X has the basis $a^1, a^2, \cdots, a^n$, and if $a^\rho a^\sigma=\sum_\tau \gamma_\tau^{\rho\sigma}a^\tau$, the formal products $a^{\rho\sigma}=a^\rho a^\sigma$ are the basis of an algebra X', called a duplicate of X, having the multiplication table $a^{\rho\sigma}a^{\theta\varphi}=\sum_\tau \gamma_\tau^{\rho\sigma}\gamma_\tau^{\theta\varphi}a^{\tau\tau}$. There may be various duplicates according as $a^{\rho\sigma}$ is identified with $a^{\tau\tau}$ for some or all distinct pairs of superscripts. The correspondence $a^{\rho\sigma}\to a^\rho a^\sigma$ determines a homomorphism $X'\sim X$. Those elements of X' which correspond to the 0 of X form an ideal O which is a left and right annihilator of X', and $X'/O\cong X^2$. There are other miscellaneous results. *C. C. MacDuffee* (New York, N. Y.).

Whitehead, J. H. C. Note on linear associative algebras. J. London Math. Soc. **16**, 118–125 (1941). [MF 5133]

Let $A=(e_1, \cdots, e_r, u_1, \cdots, u_n)_F$ be an associative algebra over the field F whose radical $N\neq 0$ is given by $(u_1, \cdots, u_n)_F$. Suppose that $e_\lambda e_\mu=\sum_{\sigma=1}^r \gamma_{\lambda\mu}^\sigma e_\sigma+\sum_{i=1}^n c_{\lambda\mu}^i u_i$. The author gives a new proof that A is isomorphic to the direct sum of the difference algebra $A-N$ and the radical N provided $|\sum_{\sigma,\rho=1}^r \gamma_{\sigma\lambda}^\sigma\gamma_{\rho\mu}^\rho|\neq 0$. The proof involves straightforward consequences of the associativity conditions and simple facts of tensor algebra. It is an extension of the author's demonstration of the corresponding theorem for Lie algebras. Moreover, new proofs of the necessary and sufficient conditions for semi-simplicity are given. *O. F. G. Schilling.*

Comét, Stig. Note sur la représentation régulière d'une algèbre linéaire. Proc. [Förhandlingar] Roy. Physiographic Soc. Lund **11**, no. 10, 5 pp. (1941). [MF 5552]

Let $\mathfrak{A}$ be a linear associative algebra over an arbitrary field. The author proves: (a) If the matrices R_i of the first regular representation are linearly dependent, the adjunction of a right unit element extends $\mathfrak{A}$ to an algebra $\mathfrak{A}'$ whose first matrices R_i' are linearly independent. (b) In order that the R_i be linearly independent, it is necessary and sufficient that $\mathfrak{A}$ contain a bilateral ideal $\mathfrak{a}\neq 0$ such that $\mathfrak{a}\mathfrak{A}=0$. (c) Unless every element of $\mathfrak{A}$ is a left divisor of zero, $\mathfrak{A}$ contains a left unit element. Result (b) is a special case of a theorem proved by the reviewer [Bull. Amer. Math. Soc. **35**, 347 (1929)]. *C. C. MacDuffee.*

Hooke, Robert. Finite groups and restricted Lie algebras. Duke Math. J. **8**, 533–540 (1941). [MF 5189]

Zassenhaus [Abh. Math. Sem. Hansischen Univ. **13**, 1–100 (1939)] has recently given a method of associating with any group G a restricted Lie algebra L over the field of residues mod p and has shown that L is nilpotent if G is finite and has the same order as G if G is a finite p-group. The present author proves the theorem that, if G is finite and Z_μ is the first term of the lower central series of G such that $Z_\mu=Z_{\mu+1}$, then the restricted Lie algebra of G is the same as that of the p-Sylow group of G/Z_μ. Thus the Lie algebras obtained by Zassenhaus' construction from finite groups may be obtained from p-groups. An example of a nilpotent restricted Lie algebra that is not obtainable from any group and examples of non-isomorphic p-groups that have the same restricted Lie algebras are given. *N. Jacobson* (Chapel Hill, N. C.).

Jacobson, N. Classes of restricted Lie algebras of characteristic p. I. Amer. J. Math. **63**, 481–515 (1941). [MF 4674]

Let A be an associative algebra over a field F and define a product operation for linear subspaces L of A by $[a, b]=ab-ba$, where ab is the associative product operation in A. Then a closed space L is a Lie algebra and one such class of Lie algebras is given by $L=A$. Another arises if A has an involution J and L is the set of J-skew quantities of A. In the case where F is non-modular these give almost all simple Lie algebras. The study is now extended to the case F of characteristic p by the introduction of the fundamental concept of restricted Lie algebras, that is, algebras L closed also with respect to a new abstract operation $a^{[p]}$. This operation is precisely that of ordinary associative pth power in the case where L is connected as above with an associative algebra. However, the algebras defined above do not, in the characteristic p case, constitute almost all simple Lie algebras. Isomorphisms, automorphisms and algebras of derivations of restricted Lie algebras are also studied fully. *A. A. Albert* (Chicago, Ill.).

Jacobson, N. Restricted Lie algebras of characteristic p. Trans. Amer. Math. Soc. **50**, 15–25 (1941). [MF 4866]

A restricted Lie algebra of characteristic p is defined as a Lie algebra in which an operation $a\to a^{[p]}$ exists, and satisfies three identities (discovered by the author) which are true in any associative algebra if $a^{[p]}$ is the pth power of a and $[ab]=ab-ba$. The author first shows, by arguments involving a construction of the reviewer [Ann. of Math. (2) **38**, 326–332 (1937)] and Witt [J. Reine Angew. Math. **177**, 152–160 (1937)], that every restricted Lie algebra L can be realized isomorphically by elements of an associative algebra A, and therefore can be represented by matrices. Secondly, if L has a finite basis, then A has a finite basis, and L has only a finite number of unequivalent irreducible representations. Thirdly, if L has finite basis and all its elements are nilpotent, then A is nilpotent [cf. also M. Zorn, Bull. Amer. Math. Soc. **42**, 485 (1936)]. Many other results are given. *G. Birkhoff* (Cambridge, Mass.).

Loonstra, F. Die Lieschen Fundamentalsätze in bewerteten Körpern. Nederl. Akad. Wetensch., Proc. **44**, 568–576 (1941). [MF 4985]

Let K be a complete valued field of characteristic 0. The author defines a K-group G as a group in which coordinates $(a_1, \cdots, a_r)$, a in K, are defined so that, if $a=(a_1, \cdots, a_r)$, $b=(b_1, \cdots, b_r)$, $c=(c_1, \cdots, c_r)$ and $c=ab$, then $c_i=\varphi_i(a, b)$, where the φ are analytic in the a's and the b's. He considers a realization of G by analytic transformations $f_i(x_1, \cdots, x_n;$

$a), i=1, \cdots, n$, that is, $f(f(x, a), b)=f(x, \varphi(a, b))$. The three fundamental theorems of Lie are proved for these realizations. *N. Jacobson* (Chapel Hill, N. C.).

Loonstra, F. Bericht über die Konstruktion und die Fortsetzung von Bewertungen. I. Nederl. Akad. Wetensch., Proc. **44**, 700–710 (1941). [MF 5116]

This paper is a review of the older literature on the construction and extension of valuations of fields.

N. Jacobson (Chapel Hill, N. C.).

Loonstra, F. Folgen und Reihen in bewerteten Körpern. I. i. Nederl. Akad. Wetensch., Proc. **44**, 286–297 (1941). [MF 5011]

This is the first of a series of papers extending elementary analysis, particularly the theory of analytic functions (defined by power series), to arbitrary complete valued fields K. The derivative of a function with range and values in K is defined. As is proved by an example uniform convergence of a sequence of functions is not a sufficient condition for the interchange of the limit and differentiation operations. The interchange, however, is permissible for the sequence of partial sums of an analytic function. The statement that absolute convergence is necessary and sufficient for ordinary convergence of a series in the non-Archimedean case is false as is seen by the example $\sum u_i$, $u_1=\cdots=u_p=p$, $u_{p+1}=\cdots=u_{p+p^2}=p^2$, $u_{p+p^2+1}=\cdots=u_{p+p^2+p^3}=p^3$, $\cdots$, in the field of p-adic numbers. *N. Jacobson* (Chapel Hill, N. C.).

Loonstra, F. Folgen und Reihen in bewerteten Körpern. I. ii. Nederl. Akad. Wetensch., Proc. **44**, 397–408 (1941). [MF 5017]

[Cf. the preceding and the following reviews.] In this paper the author obtains convergence criteria for series with terms in a perfect valued field. The product of series is defined as usual and the theorem that this series converges when one of the original series converges and the other converges absolutely is proved. Radius of convergence and majorants of power series are considered. *N. Jacobson.*

Loonstra, F. Folgen und Reihen in bewerteten Körpern. II. Nederl. Akad. Wetensch., Proc. **44**, 577–589 (1941). [MF 4976]

[Cf. the two preceding reviews.] The topics considered are (1) substitution of one series in another, (2) implicit functions defined by $F(x, y)=0$, where F is a power series in x and y, and (3) analytic continuation. The usual theorems of real analysis hold for (1) and (2). These are derived by the method of majorant series. The ordinary theory of (3) breaks down completely in the p-adic case for no analytic continuation is possible here. An essential point of the proof of this fact is the non-Archimedean property of the metric, namely, $|x_0|<R$, $|x-x_0|<R$ implies $|x|<R$.

N. Jacobson (Chapel Hill, N. C.).

Loonstra, F. Die Lösung von Differentialgleichungen in einem bewerteten Körper. Nederl. Akad. Wetensch., Proc. **44**, 409–419 (1941). [MF 5018]

This paper considers existence theorems for ordinary and partial differential equations for functions in a complete valued field of characteristic 0. Since a definition of the definite integral is lacking in the non-Archimedean case, the author restricts his attention to analytic functions. The usual proofs using majorant series carry over.

N. Jacobson (Chapel Hill, N. C.).

Deuring, Max. Die Typen der Multiplikatorenringe elliptischer Funktionenkörper. Abh. Math. Sem. Hansischen Univ. **14**, 197–272 (1941). [MF 5445]

This is an investigation of the exact structure of the "complex multiplications" of an elliptic function field K in the case when the field k of constants is not the classical field of complex numbers, but is a field of prime characteristic p. These multiplications are defined in terms of meromorphisms. A meromorphism μ of K is an isomorphism which leaves all constants fixed and which maps K onto a subfield of itself. All such μ can be combined from automorphisms of K and meromorphisms "normalized" with respect to a fixed prime divisor (corresponding to a fixed choice of the point at ∞). The normalized meromorphisms form a ring R; addition is performed by multiplying certain corresponding divisor classes, multiplication by multiplying the isomorphisms. R contains the ring of all rational integers ("natural" multiplications), and is an integral domain. Hasse has shown [J. Reine Angew. Math. **175**, 55–62, 69–88, 193–208 (1936)] that the quotient field of R is either (1) the rational numbers, (2) an imaginary quadratic number field F or (3) a definite quaternion algebra Q. Deuring completely determines the structure of R: (1) R is the ring of all integers; (2) F is a field in which the characteristic p of K is a product of two distinct ideals, and R is an order in F with conductor prime to p; (3) Q is ramified at p and ∞, and R is a maximal order in Q. Furthermore, each of these cases is possible, and Deuring determines for given R exactly how many values are possible for the (usual) invariant j of K. For types (2) and (3) j is absolutely algebraic (over the prime field of K), and for type (3) it has at most the degree 2. Tables of these invariants are given. Type (3) does not arise in the classical theory, and it is present here if and only if the given field K has no divisor classes of order p. The proofs of these results depend on the standard method of representing multiplications by matrices, as developed for the general case by Deuring [J. Reine Angew. Math. **183**, 25–36 (1940); these Rev. **2**, 246] on the relation between multiplications of a field and those of one of its subfields, and on a transition from an elliptic field K of characteristic 0 to one of characteristic p, found by reducing the coefficient field of K modulo a suitable prime divisor. Relations of the results to class field theory and to the class invariants of an imaginary quadratic field will be studied in a later paper. *S. Mac Lane* (Cambridge, Mass.).

ANALYSIS

Fourier Series and Generalizations, Integral Transforms

Ghizzetti, Aldo. Sui coefficienti di Eulero-Fourier di una funzione limitata. Ann. Scuola Norm. Super. Pisa (2) **9**, 215–223 (1940). [MF 5463]

Let $f(t)$ be a function of period 2π, such that $0 \leqq f(t) \leqq 1$, and let a_n, b_n be its Fourier coefficients. The object of the paper is to establish best possible inequalities which must be satisfied by a_n, b_n when the coefficients $a_0, a_1, b_1, \cdots, a_{n-1}, b_{n-1}$ are supposed known. Explicit inequalities are established for a_1, b_1 (a_0 being given) and for a_2, b_2 (a_0, a_1, b_1 being given). [See, in this connection, the results obtained by B. Friedman, Bull. Amer. Math. Soc. **47**, 84–92 (1941); these Rev. **2**, 279.] *R. Salem* (Cambridge, Mass.).

Bhatnagar, S. P. On the Fourier coefficients of a discontinuous function. Proc. Edinburgh Math. Soc. (2) 6, 231–256 (1941). [MF 5864]

The main results of the paper concern the interdependence of the behavior of an integrable and periodic function f in the neighborhood of a point x and the Cesàro summability of the sequence $n(b_n \cos nx - a_n \sin nx)$, where a_n and b_n are the Fourier coefficients of f. *A. Zygmund.*

Chow, Hung Ching. On a theorem of O. Szász. J. London Math. Soc. 16, 23–27 (1941). [MF 5900]

O. Szász [Duke Math. J. 4, 401–407 (1938)] and G. Maruyama [Tôhoku Math. J. 46, 68–74 (1939); these Rev. 1, 138] have established formulae connecting the jump of a function $f(x)$ with the Fejér sums of the series conjugate to the Fourier series of $f(x)$. The purpose of the present paper is to extend these formulae to the Cesàro sums of any order $\alpha > 0$ of the conjugate series. *R. Salem.*

Foà, Alberto. Aggiunta alla nota: "Sulla sommabilità assoluta $|C, \alpha|$ delle serie di Fourier." Boll. Un. Mat. Ital. (2) 3, 393–394 (1941). [MF 5709]

The author corrects a theorem of his earlier paper [Boll. Un. Mat. Ital. (2) 2, 325–332 (1940); cf. these Rev. 2, 94] by adding the following hypothesis in the case $\alpha > 1/p$: the function

$$\int_\epsilon^t (t-u)^{\alpha-1}\{f(x_0+u)+f(x_0-u)\}\,du$$

is of bounded variation in the interval (ϵ, π), where x_0 denotes the point at which the absolute summability of the Fourier series is considered. *A. Zygmund.*

Marcinkiewicz, J. and Zygmund, A. On the behavior of trigonometric series and power series. Trans. Amer. Math. Soc. 50, 407–453 (1941). [MF 5719]

The paper is devoted to the further application of an idea of B. Kuttner [J. London Math. Soc. 10, 131–140 (1935)] which in a previous paper [Fund. Math. 26, 1–43 (1936)] the authors had already developed into a fruitful instrument in the theory of trigonometric series. Let $\sigma_n^\alpha(\theta)$ denote the Cesàro means of order α of (1) $\frac{1}{2}a_0+\sum(a_\nu \cos \nu x + b_\nu \sin \nu x)$ and $\tilde{\sigma}_n^\alpha(\theta)$ those of (2) $\sum(a_\nu \sin \nu x - b_\nu \cos \nu x)$. Further, let $\lim_n \inf \sigma_n^\alpha(\theta) = \sigma_\alpha(\theta)$, $\lim_n \sup \sigma_n^\alpha(\theta) = \sigma^\alpha(\theta)$, $\sigma^\alpha(\theta) - \sigma_\alpha(\theta) = \omega_\alpha(\theta)$ and a similar notation for the conjugate series.

Theorem 1. Suppose that the series (1) is summable A (Abel) to sum $S(\theta)$ at every point of a set E of positive measure, and that $\alpha > -1$, $\sigma_\alpha(\theta) > -\infty$ for $\theta \in E$. Then at almost every point of E we have the following relations: $\sigma^\alpha(\theta) < \infty$, $-\infty < \tilde{\sigma}_\alpha \leqq \tilde{\sigma}^\alpha < +\infty$, $\tilde{\omega}_\alpha(\theta) = \omega_\alpha(\theta)$. Moreover the series (1) and (2) are summable $(C, \alpha+\epsilon)$ at almost every point of the set E, and $S(\theta) = \frac{1}{2}|\sigma_\alpha(\theta)+\sigma^\alpha(\theta)|$, $\tilde{S}(\theta) = \frac{1}{2}|\tilde{\sigma}_\alpha(\theta) + \sigma^\alpha(\theta)|$ almost everywhere in E, where $\tilde{S}(\theta)$ denotes the $(C, \alpha+\epsilon)$ sum of (2). Theorem 2. Suppose that the αth Cesàro means (where $\alpha > -1$) of (1) are bounded at every point of a set E of positive measure, that is, that $\sigma_n^\alpha(\theta) = O(1)$ for every $\theta \in E$. Then at almost every point of E the series (1) and (2) are summable $(C, \alpha+\epsilon)$ $(\epsilon > 0)$ and have the relations stated in the previous theorem.

Both theorems have been proved in the previous paper of the authors. The proof of the first theorem is here substantially the same. The proof of the second is far more difficult than the first. The chief simplification of its new proof in comparison with earlier ones is that the theory of higher generalized derivatives is not required. The proof is based on the application of summability (C^*, α) [Plessner] and a result of Privaloff: (1) is summable (C^*, α) at the point θ_0 to sum S, if $\sigma_n^\alpha(\theta_0+h_n) \to S$ for every sequence $h_n = O(1/n)$. The required Privaloff result reads: Let $F(z) = u(r, \theta) + iv(r, \theta)$ be a function regular inside the unit circle, and let us suppose that there is a set E of positive measure situated on the circumference $|z| = 1$ and such that for any $z_0 \in E$ the function $u(r, \theta)$ is bounded when z tends to z_0 along non-tangential paths. Under these conditions, for almost every point $z_0 \in E$, $u(r, \theta)$ and $v(r, \theta)$ tend to finite limits when z tends to z_0 along any non-tangential paths.

The second part of the paper is entirely new. A closed set L is said to be of circular structure with center z_0 if with $z_1 \in L$ also $E_z[|z-z_0| = |z_1-z_0|] \in L$. Theorem A. Suppose that the partial sums of the series $\sum_0^\infty c_\nu z^\nu$ are bounded at every point of a set E situated on the circumference of the unit circle and let $t(z)$ denote the $(C, 1)$ sum of the series. Then for almost every $z \in E$, the set $L(z)$ of limit points of the partial sums of the series is of circular structure, with center $t(z)$. Later it is shown that, for $c_\nu \to 0$, $L(z)$ is an annulus. The result is generalized to (C, α) sums with $\alpha > -1$. The proof uses Kuttner's idea. On the same lines a result about the strong summability of the conjugate series is proved. In the last section analogous results for trigonometrical integrals in the Fourier-Stieltjes form are enunciated without proofs. *F. Wolf* (St. Paul, Minn.).

Salem, R. On trigonometrical series whose coefficients do not tend to zero. Bull. Amer. Math. Soc. 47, 899–901 (1941). [MF 5934]

Let $\sum \rho_n \cos(nx - \alpha_n)$ be a trigonometric series such that $\rho_n \geqq 0$, $\limsup_{n\to\infty} \rho_n > 0$. It is shown that, if

$$\lim_{n\to\infty} \rho_n \cos(nx - \alpha_n) = 0$$

for x belonging to a perfect set P which contains the point 0, and if $F(x)$ is any non-decreasing function constant in the intervals contiguous to P, then

$$\limsup_{n\to\infty} \int_0^{2\pi} \cos 2nx\, dF(x) = F(2\pi) - F(0).$$

A. Zygmund (Ann Arbor, Mich.).

Salem, R. On some properties of symmetrical perfect sets. Bull. Amer. Math. Soc. 47, 820–828 (1941). [MF 5496]

Let $\xi_1, \xi_2, \cdots, \xi_p, \cdots$ be a sequence of numbers satisfying the conditions $0 < \xi_p < \frac{1}{2}$ for every p, and let E be the perfect set constructed on the interval $(0, 2\pi)$ in the well-known Cantor way, where, however, at the pth stage we divide the intervals that remain into subintervals of lengths proportional to ξ_p, $1-2\xi_p$, ξ_p and remove the central part. Hence the measure $E(p)$ of the pth approximation of E is $2^{p+1}\pi\xi_1\xi_2 \cdots \xi_p$. Let $F(x)$ be the non-decreasing function, continuous, constant over any interval outside E, and constructed in the usual way. It was Menchoff who first pointed out [C. R. Acad. Sci. Paris 163, 433–436 (1916)] that the Fourier-Stieltjes coefficients

$$c_n = (2\pi)^{-1} \int_0^{2\pi} e^{-inx} dF(x)$$

of the function F may tend to 0 as $n \to +\infty$. [He considered the case $1-2\xi_p = 1/(p+1)$.] Without essentially changing Menchoff's argument, Salem shows that Menchoff's result holds if $E(p)$ is a convex function of p and $E(p+1)/E(p)$

tends to 1 as $p\to\infty$. He also shows that the series

$$\sum_{n=1}^{\infty} E^2([\alpha \log n])|c_n|^2$$

is convergent if only $\alpha>3/(2\log 2)$. A set S situated on the interval $(0, 2\pi)$ is said to be of type N if there exists a trigonometrical series absolutely convergent in S but not everywhere. The author shows that the sum of a set of type N and of any finite set is still of type N. [Marcinkiewicz showed [Trav. Soc. Sci. Lettres Wilno, Cl. Sci. Math. Nat. 12, 1–6 (1937)] that the sum of two sets of type N may be the whole interval $(0, 2\pi)$, and so not N.] *A. Zygmund.*

Menchoff, D. Sur la représentation des fonctions mesurables par des séries trigonométriques. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 667–692 (1941). (French. Russian summary) [MF 5503]

The paper contains a detailed proof of a result announced earlier [Rec. Math. [Mat. Sbornik] N.S. 8 (50), 493–518 (1940); these Rev. 2, 189], to the effect that for every measurable and finite function $f(x)$ defined in the interval $(0, 2\pi)$ there exists a trigonometric series convergent to $f(x)$ almost everywhere (the series is not unique).

A. Zygmund (South Hadley, Mass.).

Menchoff, D. Sur la convergence uniforme des séries de Fourier. C. R. (Doklady) Acad. Sci. URSS (N.S.) 32, 245–246 (1941). [MF 5861]

The author states, with a brief sketch of proof, the following theorem, which generalizes his own earlier result [Rec. Math. [Mat. Sbornik] N.S. 8 (50), 493–518 (1940); cf. these Rev. 2, 189]. To every measurable function $f(x)$ defined in the interval $0\leqq x\leqq 2\pi$, and finite almost everywhere there, and to every $\sigma>0$, there exists a continuous function $G(x)$, $0\leqq x\leqq 2\pi$, which coincides with $f(x)$ except for a set of measure less than σ and has its Fourier series uniformly convergent. *A. Zygmund* (Ann Arbor, Mich.).

Lévine, B. and Lifschetz, M. Quasi-analytic functions represented by Fourier series. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 693–711 (1941). (Russian. English summary) [MF 5504]

Notations: $f(x)$ being integrable L in $[-\pi, \pi]$,

$$f(x)\sim a_0/2+\sum (a_{n_k}\cos n_k x+b_{n_k}\sin n_k x);$$

σ denotes the exponent of convergence of $\{n_k\}$;

$$\phi(\alpha)=\int_{\pi-\alpha}^{\pi}|f(x)|\,dx$$

(integral order of $f(x)$ at $x=\pi$),

$$\rho=\overline{\lim_{\alpha\to+0}}\frac{\log(-\log\phi(\alpha))}{-\log\alpha}$$

($x=\pi$ is a zero of exponential order ρ of $f(x)$). The reviewer proved that, if $\sigma<1$, $\rho>\sigma/(1-\sigma)$, then $f(x)\equiv 0$ (almost everywhere) [Séries de Fourier et classes quasi-analytiques de fonctions, Paris, 1935]. Later N. Wiener and the reviewer gave a new proof of this theorem. They also proved that, if $\rho=\sigma/(1-\sigma)$, then there exists a $f(x)\not\equiv 0$ with the given ρ and σ [Mandelbrojt and Wiener, C. R. Acad. Sci. Paris 203, 34–36, 233–234 (1936)]. The authors of the present paper give a comprehensive generalization of the cited theorem, with a very interesting proof. $N(t)$ denotes the number of integers n_k with $n_k<t$. Putting

$$y=-(\log\phi(\alpha))/\alpha,$$

$\eta(y)$ denotes α as a function of y;

$$\zeta(y)=2y^2\int_0^{\infty}\frac{N(t)}{t}\frac{dt}{t^2+y^2}.$$

Their fundamental theorem is: (I) If (1) $\underline{\lim}_{y\to\infty}[3\zeta(2y)-y\eta(y)]=-\infty$, $f(x)\equiv 0$ almost everywhere. The proof amounts to this: Putting

$$F(z)=\int_{-\pi}^{\pi}f(t)e^{-izt}dt,\quad \Phi(z)=\frac{F(z)Q(z)}{\sin\pi z},\quad \theta(z)=\Phi(z)\Phi(\omega z)\Phi(\omega^2 z)$$

with $\omega=e^{2\pi i/3}$, the inequality $|\theta(z)|<ce^{3\pi(2y)-y\eta(y)}$ holds on the boundary of every triangle with vertices at $-i|z|$, $-i\omega|z|$, $-i\omega^2|z|$, c being a constant, y the apothem of the triangle. From (1) it follows that $\theta(z)\equiv 0$, $F(z)\equiv 0$ and $f(x)\equiv 0$ almost everywhere. This is in particular true if $\underline{\lim}\,[12\zeta(y)-y\eta(y)]=-\infty$. (II) If $\sigma(t)$ is such that $\sigma(t)\to\sigma<1$, $t\sigma'(t)\log t\to 0$ (when $t\to\infty$), then there exists a $f(x)\not\equiv 0$ with $N(t)\sim t^{\sigma(t)}$, and such that $\underline{\lim}\,[c_1\zeta(y)-y\eta(y)]=-\infty$, $c_1>\frac{1}{2}-\varepsilon$. If in (I) $N(t)<c_1t^{\rho+\varepsilon}+c_2$, $\log\phi(\alpha)<-\alpha^{-\rho+\varepsilon_1}$ for an infinity of $\alpha\to+0$, then it becomes the reviewer's theorem. (II) proves that, in the latter theorem, the inequality $\rho>\sigma/(1-\sigma)$ cannot be replaced by the equality $\rho=\sigma/(1-\sigma)$. But if this equality holds and if $\log\phi(\alpha)<-A\alpha^{-\rho}$ for an infinity of $\alpha\to 0$, and if $N(t)<Bt^{\rho}$, then $f(x)\equiv 0$, if $A^{1/(1+\rho)}>12\pi B/\sin(\pi\rho/2)$, and there exists a $f(x)\not\equiv 0$, with $\rho=\sigma/(1-\sigma)$, $A^{1/(1+\rho)}>\frac{1}{2}\pi B/\sin(\pi\rho/2)$. This gives a significant sharpening of the reviewer's theorem. It seems to the reviewer that, in the analysis which precedes, $\zeta(y)$ has to be defined by

$$\log y+2y^2\int_0^{\infty}\frac{N(t)}{t}\frac{dt}{t^2+y^2},$$

$Q(z)$ having to be replaced by $z\Pi(1-z^2/n_k^2)$. The authors give also a theorem which combines a theorem of de la Vallée Poussin (condition B) [see the book mentioned above] with the reviewer's theorem already cited (condition A). If

$$f(x)\sim\tfrac{1}{2}a_0+\sum(a_n\cos nx+b_n\sin nx),$$

and $\log\phi(\alpha)<-\alpha^{-\rho+\varepsilon}$ for α sufficiently small, if the sequence $\{n\}$ can be divided in two subsequences $\{n_k\}$, $\{m_i\}$ such that the exponent of convergence of $\{n_k\}$ is $\sigma<1$, if (A) $\rho>\sigma/(1-\sigma)$, (B) $|a_{m_i}|+|b_{m_i}|<e^{-\gamma(m_i)}$ $(i\geqq 1)$, with $t\gamma'(t)/\gamma(t)\geqq\sigma_1>\sigma(t>t_0)$ and $\int^{\infty}\gamma(t)dt/t^2=\infty$, then $f(x)\equiv 0$ almost everywhere. The authors transfer these results to the expansions of a function in a series of fundamental functions of certain differential systems.

S. Mandelbrojt (Houston, Tex.).

Levine, B. and Lewitan, B. Supplement to a note "On Fourier series of generalized almost periodic functions." Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 109–110 (1940). (Russian) [MF 4752]

In addition to a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 22, 539–542 (1939)], the authors show that for any given regular method of summability it is possible to construct a generalized almost periodic function which has no generalized mean. *J. D. Tamarkin.*

Mazur, S. et Orlicz, W. Sur quelques propriétés de fonctions périodiques et presque-périodiques. Studia Math. **9**, 1–16 (1940). (French. Ukrainian summary) [MF 5252]

A known theorem of Cantor-Lebesgue-Steinhaus states that for any real constants θ_n the relation

$$\overline{\lim}_{n\to\infty} |\cos(nx+\theta_n)| = 1$$

holds for almost all x. The authors find the following generalization. If $f(x)$ is a periodic measurable function then for $\omega_n \to \infty$ and any θ_n the relation

$$\overline{\lim_{n\to\infty}} f(\omega_n x+\theta_n) = \text{ess. l.u.b. } f(x)$$

holds for almost all x. They also extend the result to almost periodic functions, but only to those of Bohr's type. They have other results concerning sums and products of functions with independent frequencies and they also comment on a conjecture of Steinhaus that for any sequence of functions $f_n(x)$ of period 1 which are uniformly bounded the relation $\overline{\lim}_{n\to\infty} f_n(nx) = \text{const.}$ holds almost everywhere.

S. Bochner (Princeton, N. J.).

Kovanko, A. S. Sur la compacité des systèmes de fonctions presque périodiques généralisées de A. S. Besicovitch. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 117–118 (1941). [MF 5855]

The author gives necessary and sufficient conditions for the B_w compactness of a set of B_w almost periodic functions.

R. H. Cameron (Cambridge, Mass.).

Assadourian, Fred. Intransitive Abelian almost-translation groups of almost-periodic functions. Duke Math. J. **8**, 518–524 (1941). [MF 5187]

Let $f(t) = (f_1(t), \cdots, f_m(t))$ be a vector whose components are almost periodic functions, let S be a permutation of the indices $1, \cdots, m$ and let $Sf(t)$ be the vector whose components are permuted accordingly. H. Bohr and D. A. Flanders [Danske Vid. Selsk. Math.-Fys. Medd. **15**, 1–49 (1937)] have investigated the case in which $Sf(t)$ is the uniform limit, as $k\to\infty$, of the sequence $f(t+t_k)$ for some suitable translations $\{t_k\}$, and, if there exists a transitive Abelian group $G = (S_1, \cdots, S_m)$ of such permutations, the Fourier series of $f_h(t)$, $h = 1, \cdots, m$, have the form $\sum_n \chi_n(S_h) a_n e^{\Omega_n t}$, where $\chi_1(S), \cdots, \chi_n(S), \cdots$ is a sequence of characters of the group G with suitable properties. The author gives an exhaustive extension of the previous result to the case in which the group G is not necessarily transitive.

S. Bochner (Princeton, N. J.).

Trejo, César A. On functions with quasi-periodic local mean. An. Soc. Ci. Argentina **132**, 137–138 (1941). (Spanish) [MF 5732]

Given a function $g(x)$ of $L(-\infty, \infty)$, the author studies the class of bounded measurable functions $f(x)$ such that the convolution

$$F(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

is uniformly almost periodic. A sufficient condition for every such $f(x)$ to have a generalized Fourier series is that the Fourier transform of $g(x)$ should have no real zeros. This condition is, however, not necessary, as is shown by the example of the function $g(x) = 1/(2\delta)$ if $|x| < \delta$, $g(x) = 0$ elsewhere. In this case $f(x)$ has an almost periodic "local mean," $f_h(x) = (1/(2\delta))\int_{x-\delta}^{x+\delta} f(t)dt$, and $f(x)$ still has a generalized Fourier series. Proofs are to appear later.

R. P. Boas, Jr. (Durham, N. C.).

Potter, H. S. A. The mean values of certain Dirichlet series, II. Proc. London Math. Soc. (2) **47**, 1–19 (1940). [MF 3648]

Let ϕ be a function representable by an ordinary Dirichlet series of signature $\{\lambda, k, \gamma\}$, $\lambda > 0$, $k > 0$, $\gamma = \pm 1$. The author discusses the asymptotic value of $I(T) = \int_0^T |\phi(\sigma+it)|^2 dt$ for the critical line $\sigma = \frac{1}{2}k$: If $\sum |a_n|^2$, over $n < x$, is asymptotic to $x^k L(x)$, where $L(x)$ is a slowly increasing function, such that $L_1(x) = \int_1^x L(t)t^{-1}dt \to \infty$ as $x \to \infty$, then $I(T) \sim 2kTL_1(T)$. Similar results are obtained for a wider class of Dirichlet series. The behavior of $I(T)$ for $\sigma \neq k/2$ was discussed by the same author [Proc. London Math. Soc. (2) **46**, 467–478; these Rev. **2**, 95].

F. Bohnenblust (Princeton, N. J.).

Greenwood, R. E. Hankel and other extensions of Dirichlet's series. Ann. of Math. (2) **42**, 778–805 (1941). [MF 4972]

The author considers what he calls the modified exponential series

$$g(s) = \sum_1^{\infty} a_n e^{-\lambda_n s} G(\lambda_n s),$$

where (1) $G(z)$ is holomorphic in $\Re(z) > 0$, (2) $|G(z)| \leqq K(x_0)$, $\Re(z) \geqq x_0 > 0$, and (3) $G(z) \sim \sum_0^{\infty} b_n z^{-n}$, $b_0 \neq 0$, for large z. If the associated Dirichlet series $\sum a_n e^{-\lambda_n s}$ has a non-negative abscissa of convergence, then the two series are equiconvergent with respect to ordinary, uniform and absolute convergence. The exponential series vanishes identically if $g(s)$ has infinitely many zeros in a sector of convergence. The author gives two inversion formulas of which one reads

$$(1/2\pi i)\int_{c-i\infty}^{c+i\infty} g(s)e^{\omega s}ds/s = \sum_1^n a_m(1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{(\omega-\lambda_m)s}G(\lambda_m s)ds/s,$$

where c exceeds the abscissa of convergence and $\lambda_n < \omega < \lambda_{n+1}$. A special case is the Hankel series

$$h(s) = \sum_1^{\infty} a_n (i\lambda_n s)^{\frac{1}{2}} H_\nu^{(1)}(i\lambda_n s),$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind of order ν. The author shows that, for real $\nu \geqq \frac{1}{2}$, $a_n \geqq 0$, the intersection of the line of convergence with the real axis is a singular point of $h(s)$.

E. Hille (Palo Alto, Calif.).

Kac, M. Convergence and divergence of non-harmonic gap series. Duke Math. J. **8**, 541–545 (1941). [MF 5190]

Let $\lambda_1, \lambda_2, \cdots$ be a sequence of real numbers such that $\lambda_{k+1}/\lambda_k > q$, and $a_1, a_2, \cdots$ a sequence of complex numbers The author proves that the series (1) $\sum a_k e^{\Omega_k t}$ converges almost everywhere, $-\infty < t < +\infty$, if $q > 1$ and $\sum |a_k|^2 < \infty$, while the series (1) diverges almost everywhere, $-\infty < t < +\infty$, if $q > \frac{1}{2}(1+5^{\frac{1}{2}})$ and $\sum |a_k|^2 = \infty$. (The situation in the case that $1 < q \leqq \frac{1}{2}(1+5^{\frac{1}{2}})$ and $\sum |a_k|^2 = \infty$ is not settled.) The proof is an adaptation of those used for periodic gap series. These proofs are made applicable by the construction of a suitable finite, completely additive measure (which has a continuous positive density) on the axis $-\infty < t < +\infty$.

P. Hartman (Fort Sill, Okla.).

Zaanen, A. C. A theorem on a certain orthogonal series and its conjugate series. Nieuw Arch. Wiskde (2) **20**, 244–252 (1940). [MF 5209]

Consider in the interval $0 \leqq x \leqq \pi$ the following two boundary value problems:

(I) $v''(x)+\lambda v(x)=0,\ av(0)+v(\pi)=0,\ v'(0)+av'(\pi)=0,$
(II) $w''(x)+\lambda w(x)=0,\ w(0)+aw(\pi)=0,\ aw'(0)+w'(\pi)=0,$

where $a^2 \neq 0, 1$, and let $\{u_n(x)\}$ and $\{v_n(x)\}$ be the orthonormal systems of solutions of (I) and (II), respectively. If $\sum a_n v_n(x)$ is the Fourier development of a function $f(x) \in L(0, \pi)$ with respect to the system $\{u_n(x)\}$, the author calls the series $\sum a_n w_n(x)$ conjugate to $\sum a_n u_n(x)$. Let $SL(f)$ be the series $\sum a_n u_n(x)$ and $\overline{SL}(f)$ the conjugate series, and let $S(f)$ and $\bar{S}(f)$ denote, respectively, the cosine development of the function $f(x)$ and the series conjugate (in the trigonometric sense) to $S(f)$. In an earlier paper [Compositio Math. **7**, 253–282 (1939); cf. these Rev. **1**, 141], the author proved that the series $SL(f)$ and $S(f)$ are uniformly equiconvergent in every interval $(\epsilon, \pi-\epsilon)$. He now shows that in every $(\epsilon, \pi-\epsilon)$ the series $\overline{SL}(f)$ and $\bar{S}(f)$ are also uniformly equiconvergent but in the wider sense (that is, the difference $\overline{SL}(f)-\bar{S}(f)$ converges uniformly in $(\epsilon, \pi-\epsilon)$ but not necessarily to 0). *A. Zygmund.*

Hardy, G. H. Notes on special systems of orthogonal functions. IV. The orthogonal functions of Whittaker's cardinal series. Proc. Cambridge Philos. Soc. **37**, 331–348 (1941). [MF 5542]

[The third part of this series appeared in the same Proc. **36**, 1–8 (1940); cf. these Rev. **1**, 141.]

After defining completeness and closure for an orthogonal system the author defines a normal orthogonal system for the interval $(-\infty, \infty)$ by means of the equations $\phi_n(x)=X_n(x)$, $|x| \leqq a$, and $\phi_n(x)=0$, $|x|>a$, where $X_n(x)$ is a complete normal orthogonal system for the finite interval $(-a, a)$. He then remarks that the system ϕ_n is not usually complete. If ψ_n is the Fourier transform of ϕ_n the functions ψ_n also form an orthogonal set. The case $a=\pi$, $(2\pi)^{\frac{1}{2}}X_n=e^{inx}$ is discussed in detail and the associated functions

$$\psi_n=\frac{(-)^n \sin(\pi x)}{\pi}\frac{}{x-n},$$

which occur in the theory of interpolation, are shown to form a normal orthogonal system complete in the class of Paley-Wiener functions defined by

$$g(x)=(2\pi)^{-\frac{1}{2}}\int_{-a}^{a} f(t)e^{-ixt}dt,$$

where f^2 is integrable in the sense of Lebesgue. The paper contains also the evaluation of some definite integrals and the limiting forms of some finite series. A discussion is given of a certain Fourier series, a particular integral equation with infinite limits and of the cardinal series of interpolation. *H. Bateman* (Pasadena, Calif.).

Salem, Raphaël. A new proof of a theorem of Menchoff. Duke Math. J. **8**, 269–272 (1941). [MF 4591]

Let $\varphi_1(x), \varphi_2(x), \cdots$ be an orthogonal and normal system on the interval (a, b), and let $s_n(x)$ denote the nth partial sum of the series

(†) $$a_1\varphi_1(x)+a_2\varphi_2(x)+\cdots,$$

where $a_1^2+a_2^2+\cdots$ is finite. It is shown that, if $n(x)$ is any measurable function of x assuming only positive and integral values not greater than N, then

(*) $$\int_a^b s_{n(x)}(x)dx \leqq C(\sum_{r=1}^{n} a_r^2)^{\frac{1}{2}} \log N,$$

or even

(**) $$\int_a^b s_{n(x)}^2(x)dx \leqq C(\sum_{r=1}^{N} a_r^2) \log^2 N,$$

where C denotes an absolute constant. The inequality (*) is due to Menchoff [Fund. Math. **4** (1923)] and has as a simple consequence another theorem of Menchoff to the effect that the convergence of the series $\sum a_n^2 \log^2 n$ implies convergence almost everywhere of the series (†). The proof of inequalities (*) and (**) given by Salem is simple and ingenious, and is based on the use of the Bessel inequality for orthogonal functions of two variables. *A. Zygmund.*

Walfisz, Arnold. Über einige Orthogonalreihen. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 411–420 (1940). (German. Russian summary) [MF 5279]

Let I be a finite interval, and $f_1(x), f_2(x), \cdots$ a system orthonormal in I. Let $a_1, a_2, \cdots$ and $c_1, c_2, \cdots$ be two sequences of real numbers, and let

$$A_n=\sum_{kk=n} c_k a_k, \quad K_n=\sum_{kk=n} |c_k a_k|, \quad F_k(x)=\sum_{k=1}^{\infty} a_k f_{kk}(x).$$

The paper is concerned with the problem of the validity of the relation

(*) $$\sum_{k=1}^{\infty} c_k F_k(x)=\sum_{n=1}^{\infty} A_n f_n(x),$$

which formally may be derived by substituting the series defining $F_k(x)$ into the left-hand side of (*) and then inverting the order of summation. Problems of this kind, when the system $f_n(x)$ is the trigonometric system, have been considered by a number of writers, in particular by Chowla [Proc. Indian Acad. Sci. Sect. A. **5**, 511–516 (1937)] and Hartman [Amer. J. Math. **60**, 66–74 (1938)]. Walfisz shows that both sides of (*) are convergent and equal for almost every x, if

(**) $$\sum K_n^2 \log^2 n < \infty.$$

This is a generalization (take $a_1=1$, $a_2=a_3=\cdots=0$) of the well-known Rademacher-Menchoff theorem: the series $\sum c_n f_n(x)$ converges almost everywhere if $\sum c_n^2 \log^2 n < \infty$. It is also shown that the condition (**) may be replaced by $a_n=O(n^{-\frac{1}{2}} \log^{-\alpha} n)$, $c_n=O(n^{-\frac{1}{2}} \log^{-\alpha} n)$ for any $\alpha>3/2$. *A. Zygmund* (South Hadley, Mass.).

Civin, Paul. Inequalities for trigonometric integrals. Duke Math. J. **8**, 656–665 (1941). [MF 5949]

If $f(x)=\int_{-R}^{R} e^{ixt}ds(t)$ ($s(t)$ of bounded variation), and $|f(x)| \leqq M(-\infty<x<\infty)$, the author seeks bounds for $g(x)=\int_{-R}^{R}\mu(t)e^{ixt}ds(t)$, where $\mu(t)$ is a function such that for some real a the function $\mu(t)e^{iat}$ has an absolutely convergent Fourier series $\sum c_n e^{ip(n)t}$, $p(n)=n\pi/R$. He shows that $|g(x)| \leqq M\sum|c_n|$, and then applies this result to special classes of functions $\mu(t)$ for which $\sum|c_n|$ can be conveniently estimated. Examples: if $\mu(-R)=e^{i\theta}\mu(R)$ $(0 \leqq \theta<2\pi)$, $2aR=\theta$, and

$$(-1)^n\int_{-R}^{R}\mu(t)\exp\{i[a-p(n)t]\}dt \geqq 0$$

for all integers n, then $|g(x)| \leqq |\mu(R)|M$, and the inequality is best possible. This contains in particular some generalizations of S. Bernstein's inequality for the derivative of a trigonometric polynomial, originally given by Szegö [Schr. Königsberg. Gel. Ges. Naturwiss. Kl. 5, 59–70 (1928)]. If $\mu(t)$ is even and absolutely continuous and $\mu'(t)$ is positive and non-increasing in $[0, R]$, then

$$|g(x)| \leqq 2R^{-1}\int_0^R \mu(t)dt - 2\mu(0).$$

This generalizes results of Sokolov [Bull. Acad. Sci. URSS. Sér. Math. (7) 1935, 857–884]. For the fractional derivative

$$f^{(\alpha)}(x) = \int_{-R}^{R} (it)^\alpha e^{ixt} ds(t),$$

he obtains $|f^{(\alpha)}(x)| \leqq K(\alpha)M$, where $K(\alpha) < 7/\alpha$ (a smaller but more complicated $K(\alpha)$ is also given).

R. P. Boas, Jr. (Durham, N. C.).

Levitan, B. On functions with pure point spectra. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 89–101 (1940). (Russian. English summary) [MF 4734]

A continuous function $V(\alpha)$ is said to be a "pure point function" (p.p.f.) on the interval (a, b) if for every $\epsilon > 0$ and $\delta > 0$ there exist N points $\alpha_1, \alpha_2, \cdots, \alpha_N$ such that the variation of $V'(\alpha)$ is less than ϵ on every set $(a, b) - \sum_{i=1}^{N}\delta_i$, where δ_i denotes an open interval covering α_i and $\sum_{i=1}^{N}|\delta_i| < \delta$ ($|\delta_i|$ = length of the interval δ_i). The function $V(\alpha)$ is said to be a p.p.f. on $(-\infty, \infty)$ if it is a p.p.f. on every finite interval. Furthermore, $f(x)$ is defined to be a function with a pure point spectrum if its generalized Fourier transform

$$E(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\frac{e^{-i\alpha x} - L(\alpha x)}{(-ix)^2}dx,$$

where

$$L(\alpha x) = \begin{cases} 1-\alpha x, & |x| \leqq 1, \\ 0, & |x| > 1, \end{cases}$$

is a p.p.f. The main result of the paper is that every function with a pure point spectrum which is bounded and uniformly continuous on $(-\infty, \infty)$ is an almost periodic function in Bohr's sense. However, not every Bohr almost periodic function has a pure point spectrum. *M. Kac.*

Wintner, Aurel. On Fourier averages. Amer. J. Math. 63, 698–704 (1941). [MF 5617]

The main result of this paper is that, if $|f(t)|^2$ is integrable over every finite t-interval and if there exists a sufficiently small $\epsilon > 0$ such that

$$\int_{-T}^{T} |f(t)|^2 dt = O(T^{2-\epsilon}) \qquad \text{as } T \to \infty,$$

then the Fourier average

$$\lim_{T\to\infty} (2T)^{-1}\int_{-T}^{T} f(t)e^{\lambda_n t}dt$$

exists and vanishes for almost every λ. *M. Kac.*

Meijer, C. S. Eine neue Erweiterung der Laplace-Transformation. II. Nederl. Akad. Wetensch., Proc. 44, 831–839 (1941). [MF 5920]

The proofs of the two principal theorems of Part I [Nederl. Akad. Wetensch., Proc. 44, 727–737 (1941); these Rev. 3, 38] are completed here. *R. V. Churchill.*

Mohan, B. Self-reciprocal functions involving Appell's function. J. Indian Math. Soc. (N.S.) 5, 32–37 (1941). [MF 5111]

Conditions are stated under which $g(x) = \int_0^\infty P(y)f(xy)dy$ or $\int_0^\infty P(xy)f(y)dy$ is self-reciprocal in the J_λ transform; these closely resemble some previously given by the author [Quart. J. Math., Oxford Ser. 10, 252–260 (1939); these Rev. 1, 140]. Special cases are found after considerable computation to be

$$g(x) = x^{\lambda+\nu} \times F_4(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda, 1+\tfrac{1}{2}\nu; 1+\tfrac{1}{2}\nu-\tfrac{1}{2}\lambda, 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\lambda; -\tfrac{1}{4}x^2; -\tfrac{1}{4}x^2),$$

where $-1 < \nu < \frac{1}{2}$, $\nu+2 > \lambda$, $\frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}$ a non-negative integer, and a similar function also involving F_4 (which denotes Appell's generalized hypergeometric function of two variables [Bailey, Generalized Hypergeometric Series, Cambridge, 1935, Chapter 9]). *R. P. Boas, Jr.* (Durham, N. C.).

Guinand, A. P. On Poisson's summation formula. Ann. of Math. (2) 42, 591–603 (1941). [MF 4959]

The author establishes Poisson's summation formula in the form

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}{}' \tfrac{1}{2}\{f(n+)+f(n-)\} - \int_0^N f(t)dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}{}' g(n) - \int_{0+}^{N} g(t)dt\right),$$

where

$$g(x) = 2\int_{0+}^{+\infty} \cos 2\pi x t f(t)dt,$$

provided that $f(x) = f_1(x) + f_2(x) + f_3(x)$; $f_1(\infty) = f_3(\infty) = f_3(\infty) = 0$; $f_1(x)$ and $f_2(x)$ are integrals; $xf_i'(x)$ belongs to $L^{p_i}(0, \infty)$ for some p_i in $1 < p_i \leqq 2$ $(i = 1, 2)$; and $f_3(x)$ is of bounded variation in $(0, \infty)$. The method is a modification of his earlier method [Quart. J. Math., Oxford Ser. 9, 53–67 (1938)] using the Parseval theorem for Hankel transforms of order $\frac{1}{2}$; it depends on the fact that $\{[x]-x\}/x$ is self-reciprocal with respect to such transforms. Three examples of the use of the summation formula are given, including a brief derivation of the functional equation of the Riemann zeta function. The discussion is generalized to series involving primitive characters as coefficients. *R. P. Boas, Jr.*

Cossar, J. A theorem on Cesàro summability. J. London Math. Soc. 16, 56–68 (1941). [MF 5904]

If $a > 0$, $r > 0$, $p = [r]$, $f(x)$ is Lebesgue integrable over every finite interval (a, ω) and $\int_a^\infty f(x)dx$ is bounded (C, r), then the integral $\int_a^\infty f(x)\varphi(x)dx$ is uniformly summable (C, r) to $(-1)^{p+1}\int_a^\infty f_{r+1}\varphi^{(r+1)}(x)dx$ for a class of functions $\varphi(x)$ satisfying (i) $\varphi(x) \to 0$ uniformly as $x \to \infty$, (ii) $\varphi^{(r)}(x)$ is absolutely continuous, and (iii) $\int_a^\infty x^r |\varphi^{(r+1)}(x)|dx$ is uniformly convergent. Here f_{r+1} denotes a fractional integral and $\varphi^{(r+1)}$ denotes a fractional derivative. An alternative, but more restrictive, set of conditions involving only integrals and derivatives of integral order is also given. The corresponding results for series, and for integrals with integral r, were already known. *R. P. Boas, Jr.*

Achyeser, N. and Krein, M. Some remarks about three papers by M. S. Verblunsky. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 129–134 (1940). [MF 4740]

The authors point out that the three problems considered by S. Verblunsky in his papers [Proc. Cambridge Philos.

Soc. 32, 30–39, 201–211, 521–530 (1936)] were solved in some earlier work by the authors. For a unified exposition they refer to their recent book [On Some Problems of the Theory of Moments, Kharkoff, 1938 (in Russian)].

J. D. Tamarkin (Providence, R. I.).

Achyeser, N. I. Infinite Jacobi matrices and the problem of moments. Uspekhi Matem. Nauk 9, 126–156 (1941). (Russian) [MF 5080]

A concise and elegant exposition of some fundamental theorems of the theory of the problem of moments on the basis of the theory of Jacobi matrices, using only the simplest facts of the theory of operators in Hilbert spaces.

J. D. Tamarkin (Providence, R. I.).

Geronimus, J. On the character of the solution of the moment-problem in the case of the periodic in the limit associated fraction. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 203–210 (1941). (Russian. English summary) [MF 5170]

The author considers the case where the continued fraction

$$(*) \qquad \cfrac{\lambda_1}{z-\alpha_1}\Big\rfloor - \cfrac{\lambda_2}{z-\alpha_2}\Big\rfloor - \cdots$$

associated with the integral $\int_{-\infty}^{\infty} d\psi(x)/(z-x)$ is periodic in the limit, that is, $\lambda_n \to l_\nu > 0$, $\alpha_n \to a_\nu$, $n \equiv \nu \pmod{k}$, $\nu = 1, 2, \cdots, k$. Let $\bar{R}_{i-1}(z)/\bar{P}_i(z)$, $i = 1, 2, \cdots$, be the convergents of the continued fraction obtained from (*) by replacing λ_i by l_i and α_i by a_i. Let $z_1, z_2, \cdots, z_{2r}$, $r \leqq k$, be the branch points of $\{[\bar{P}_k(z) - \bar{R}_{k-2}(z)]^2 - l_1 l_2 \cdots l_k\}^{\frac{1}{2}}$, and E the sum of the closed intervals $[z_1, z_2], \cdots, [z_{2r-1}, z_{2r}]$. The author shows that the set of the points of increase of $\psi(x)$ is dense in E, and may contain no more than a denumerable set of isolated points outside of E. Discussion of transfinite diameters of various sets is essential for the author's arguments. An analogous problem is solved also for a trigonometric problem of moments.

J. D. Tamarkin.

Schwetzow, K. I. Über das Momentenproblem von Hamburger bei der zusätzlichen Bedingung, dass ein gegebenes Intervall frei von Massen sei. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 121–128 (1940). (Russian. German summary) [MF 4739]

The author considers the problem of moments

$$c_k = \int_{-\infty}^{\infty} u^k d\phi(u), \qquad k = 0, 1, 2, \cdots,$$

with the additional restriction that $\phi(u) = \text{const.}$ for $\alpha < u < \beta$. Using the results of M. Riesz and R. Nevanlinna, he finds conditions for the existence and for the uniqueness of the solution. Thus, for the existence of the solution it is necessary and sufficient that quadratic forms

$$\sum_{i,k=0}^{n} c_{i+k} x_i x_k, \quad \sum_{i,k=0}^{n-1} d_{i+k} y_i y_k, \quad d_k = c_{k+2} - (\alpha+\beta)c_{k+1} + \alpha\beta c_k,$$

should be non-negative for $n = 0, 1, 2, \cdots$, and positive if the solution is not a step function with a finite number of jumps. For the uniqueness of the solution it is necessary and sufficient that at least one of the two Hamburger problems of moments,

$$(1) \quad c_k = \int_{-\infty}^{\infty} u^k d\phi(u), \qquad (2) \quad d_k = \int_{-\infty}^{\infty} u^k d\psi(u),$$

should be determined.

J. D. Tamarkin.

Polynomials, Polynomial Approximations

Littlewood, J. E. Mathematical notes (14): "Every polynomial has a root." J. London Math. Soc. 16, 95–98 (1941). [MF 5127]

"Any proof of this theorem must appeal by the nature of the case to limit-processes lying outside algebra proper." The problem is to find a simple and intuitive proof in which the limit processes are reduced to a minimum, and the latter requirement excludes the use of circular functions. The proof is algebraic except for the statements: (i) the square root of a positive number exists; (ii) a continuous function attains its lower bound (in a closed set). The proof is by the familiar minimum principle, and for a general polynomial this method requires the solution of the equation $z^k - c = 0$. It is first shown by the minimum principle that the polynomial $z^k - c$ has a root, and then the minimum principle is used to show that every polynomial has a root.

A. C. Schaeffer (Stanford University, Calif.).

Ostrowski, Alexander. On a theorem by J. L. Walsh concerning the moduli of roots of algebraic equations. Bull. Amer. Math. Soc. 47, 742–746 (1941). [MF 5478]

Walsh [Ann. of Math. (2) 26, 59–64 (1924)] gave an important converse of a theorem of A. E. Pellet. The author remarks that Walsh's proof of his theorem contains a gap and then proceeds to establish the theorem independently by a highly instructive use of continuity arguments and Rouché's theorem.

I. J. Schoenberg (Philadelphia, Pa.).

Samuelson, Paul A. Conditions that the roots of a polynomial be less than unity in absolute value. Ann. Math. Statistics 12, 360–364 (1941). [MF 5423]

Work is based on a theorem generally attributed to A. Hurwitz [Math. Ann. 46 (1895)] which gives necessary and sufficient conditions for a polynomial equation with real coefficients to have only roots with negative real parts. (A certain set of n determinants formed from the coefficients of the equation of degree n must all be greater than 0.) It is stated that this theorem, in slightly different form, was given in 1877 by Routh. By mapping conformally by a linear transformation, which may be chosen with real coefficients, the interior of the unit circle onto the half plane to the left of the axis of imaginaries, necessary and sufficient conditions for all roots of the given equation to lie inside the unit circle are obtained at once. Indications of the importance of this question for econometric business cycle analysis are given.

A. J. Kempner (Boulder, Colo.).

Kharadze, A. Über eine Anwendung von Polynomen, die den Jacobischen analog sind. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 15–21 (1941). (Russian. German summary) [MF 5293]

The present note generalizes certain results of Tchakaloff, Favard and others regarding the ξ-problem, that is, a more precise limitation of the number ξ entering into the mean-value theorem of integral calculus, the remainder of Taylor's formula, etc. Instead of a real interval, the author considers a polygon $[\alpha^{k-1}; 1]$ formed by joining the origin in the complex plane with the points $1, \alpha, \cdots, \alpha^{k-1}$, where α is a primitive kth root of unity; the integral $\int_{-1}^{1} f(z)dz$ is replaced here by the operator

$$I^{(\alpha)} f(z)dz = F(1) + \alpha^{k-1}F(\alpha) + \cdots + \alpha F(\alpha^{k-1}), \quad F'(z) \equiv f(z).$$

From a direct study of this operator conclusions are drawn concerning the vanishing "inside" $[\alpha^{k-1};1]$ of the real polynomial $F^{(k+1)}(z)$, where $F^{(k+1)}(z)$ is real on $[\alpha^{k-1};1]$ and $I^k(1-z^k)F^{(k+1)}(z)dz=0$. (The proofs are carried out for $k=3$.) Finally it is shown that for $k=2$ we come back to a ξ-problem for definite integrals on the interval $(-1,1)$, solved by a certain class of Jacobi polynomials.

J. A. Shohat (Philadelphia, Pa.).

Schaeffer, A. C. Inequalities of A. Markoff and S. Bernstein for polynomials and related functions. Bull. Amer. Math. Soc. 47, 565–579 (1941). [MF 5048]

This address given before the American Mathematical Society reviews the literature on the subject mentioned in the title. The bibliography contains 44 items. *G. Szegö.*

Schaeffer, A. C. and Szegö, G. Inequalities for harmonic polynomials in two and three dimensions. Trans. Amer. Math. Soc. 50, 187–225 (1941). [MF 5141]

Given the real numbers $\{\lambda_i;\mu_i\}$, $i=0,1,2,\cdots,n$, and the trigonometric polynomial

$$(1)\qquad U(\varphi)=a_0+2\sum_{m=1}^{n}(a_m\cos m\varphi+b_m\sin m\varphi).$$

The authors start with the interpolation formula

$$(2)\quad G\equiv\lambda_0a_0+2\sum_{m=1}^{n}(\lambda_ma_m+\mu_mb_m)=\frac{1}{2n}\sum_{\nu=0}^{2n-1}(-1)^\nu g(\nu\pi/n)U(\nu\pi/n),$$

where

$$(3)\qquad g(\varphi)=2\sum_{m=0}^{n}{}'(\lambda_{n-m}\cos m\varphi-\mu_{n-m}\sin m\varphi)$$

($'$ means: multiply first and last term by $\frac{1}{2}$). Precise upper bounds are found for $|G|$ if $g(\nu\pi/n)\geqq 0$ and $|U(\nu\pi/n)|\leqq 1$ or $U(\nu\pi/n)\geqq 0$ $(0\leqq\nu\leqq 2n-1)$. By specifying in various manners the λ_i, μ_i, various inequalities are obtained for the two-dimensional harmonic polynomial $U(r,\varphi)$ written in the standard form

$$U(r,\varphi)=a_0+2\sum_{m=1}^{n}r^m(a_m\cos m\varphi+b_m\sin m\varphi),$$

also for its derivatives, gradient, etc. Illustration: If

$$|U(1,\nu\pi/n)|\leqq 1,\qquad 0\leqq\nu\leqq 2n-1,$$

then

$$|U(R,\nu\pi/n)+\delta R^nU(R^{-1},\nu\pi/n)|\leqq R^n+\delta,\qquad R>1;\ \delta=+1\text{ or }-1.$$

Similar results are obtained for the three-dimensional harmonic polynomial represented by means of Legendre polynomials. Applications are made to power series and to polynomials in general. *J. A. Shohat* (Philadelphia, Pa.).

Touretsky, A. Sur quelques propriétés extremales des polynomes. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 17, 45–63 (1940). (Russian. French summary) [MF 4748]

Let $\phi_n(\theta)=\sum_{k=0}^{n}(A_k\cos k\theta+B_k\sin k\theta)=A_0+s_n(\theta)$. If

$$\left|\phi_n\left(\frac{2\nu+1}{2n+1}\pi\right)\right|\leqq 1,\qquad \nu=0,1,\cdots,2n,$$

then $|\phi_n'(\theta)|<M_n\sim(2/\pi)n\log n$ [S. Bernstein and Grandjot]. The author shows that this (asymptotic) inequality cannot be improved. If $\int_0^{2\pi}s_n(\theta)d\theta=0$, $|s_n(2\nu\pi/(n+1))|\leqq 1$, $S_n(\theta)=\int s_n(\theta)d\theta$, $\int_0^{2\pi}S_n(\theta)d\theta=0$, then $|S_n(\theta)|<L_n+\epsilon_n$, with $L_n\to\pi/2$, $\epsilon_n=O(1/n^2)$ [Gronwall]. This evaluation cannot be improved. Other analogous results are given. He proves also that, if $P_n(x)=a_0x^n+\cdots+a_n$, $|P_n(x_k)|\leqq 1$, $x_k=k/n$ $(k=0,1,\cdots,n)$, $Q_{n+1}(x)=\int_0^xP_n(x)dx$, then $|Q_{n+1}(x)|<L_n$ $(0\leqq x\leqq 1)$, with $L_n=2^{n+2}/(n\log n)^2$ for n even and $L_n=2^{n+1}/(n\log n)^2$ for n odd. This cannot be improved.

S. Mandelbrojt (Houston, Tex.).

Obreschkoff, Nikola. Über die Extremwerte von Polynomen. Jber. Deutsch. Math. Verein. 51, 76–80 (1941).

Let $f(x)$ be a polynomial of degree n with real coefficients and with $n-1$ extremal points at which $f(x)=\pm 1$. A classical argument of Tchebycheff is used to represent $f(x)$ in terms of Jacobi polynomials. *A. C. Schaeffer.*

Brzecka, V. Sur un problème d'extremum. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 33–44 (1940). (Russian. French summary) [MF 4730]

Let $y_n(x)=x^n+a_1x^{n-1}+\cdots+a_n$ be any polynomial of degree n with the highest coefficient 1. The author considers the problem of finding

$$(*)\qquad I_n(a)=\min_{y_n}\int_{-1}^{1}|y_n(x)|(x+a)^{-\frac{1}{2}}dx,$$

where $a>1$ is given, and proves that $I_n(a)=2^{-n+1}[(a+1)^{\frac{1}{2}}-(a-1)^{\frac{1}{2}}]$. This result is based on an investigation by Achyeser and Krein [Comm. Soc. Math. Kharkoff (4) 14, 47–59 (1937)] who determined

$$L_n=\min_{y_n}\left[\int_{-1}^{\alpha}|y_n(x)|dx+\int_{\beta}^{1}|y_n(x)|dx\right],\qquad -1<\alpha<\beta<1.$$

The author shows that in the case when $\alpha=-\beta$ the results of Achyeser and Krein [l.c.] can be considerably simplified giving $L_{2m-1}=2^{-2m+2}(1-\beta^2)^m$, $L_{2m}=2^{-2m+1}(1-\beta)^{m+1}(1+\beta)^m$, $m=1,2,\cdots$. The result (*) can be now obtained by suitable changes of variables. *J. D. Tamarkin.*

Sheffer, I. M. Some applications of certain polynomial classes. Bull. Amer. Math. Soc. 47, 885–898 (1941). [MF 5933]

This address, delivered before the American Mathematical Society, emphasizes recent contributions of the author. He defines Appell sets, and discusses their application to the solution of linear differential equations of infinite order with constant coefficients. The case where the generating function for the equation is an entire function of exponential type is examined in detail. More general sets of polynomials, called sets of zero type, are considered; these have the property $J[P_n(x)]=P_{n-1}(x)$, where $J[y]=\sum_{n=1}^{\infty}L_{n-1}y^{(n)}(x)$, the L_n being constants. Application of sets of zero type to the solution of equations with non-constant coefficients is indicated. The problem of null-functions for a set of operators $M_n[f(x)]=\sum_{k=0}^{n}c_{nk}f^{(k)}(0)$, that is, non-zero solutions of $M_n[f(x)]=0$, is considered; a special case of this problem is Takenaka's problem of finding the largest λ such that an entire function of exponential type less than λ cannot be a null-function for $M_n[f]=f^{(n)}(a_n)$, $|a_n|\leqq 1$. For some operators M_n there is only a finite number of linearly independent null-functions of exponential type not exceeding a given σ; but, using the theory of sets of zero type, the author shows

that there exist M_n having a continuous spectrum of such null-functions. *R. P. Boas, Jr.* (Durham, N. C.).

***Lagrange, René. Polynomes et fonctions de Legendre.** Mémor. Sci. Math., fasc. 97, 83 pp. (1939). 25 fr.

In a limited space the classical properties of Legendre polynomials are given with a more detailed study of the generalized Legendre functions $P_n^m(z)$ and $Q_n^m(z)$, n and m arbitrary. The addition theorems and asymptotic formulas are dealt with in this generality. The last chapter contains a short treatment of the convergence and summability of Legendre series. The bibliography lists 50 authors and contains various misprints and mistakes. [For instance, no paper of E. Hille was published in vols. 5 and 8 of Math. Z.; obviously two papers of E. Hilb are meant.] *G. Szegö.*

Tricomi, Francesco. Sulle serie di polinomi di Legendre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 369–390 (1940). [MF 4782]

An elementary discussion of the theory of Legendre polynomials and expansions in Legendre series restricted to functions having piecewise continuous first order derivative. *E. Hille* (Palo Alto, Calif.).

Babini, José. On some properties of the derivatives and on certain primitives of Legendre polynomials. Revista Union Mat. Argentina **7**, 65–70 (1941). (Spanish) [MF 5531]

From an expansion of $D_n^{(s)} \equiv (2^n n!)^{-1} d^s(x^2-1)^n/dx^s$ the relation (*) $s!D_n^{(2n-s)} = (2n-s)!(x^2-1)^{s-n}D_n^{(s)}$ (n = natural number, $s \leqq n$) is obtained. Since $D_n^{(n+s)} = d^s P_n(x)/dx^s \equiv P_n^{(s)}$, where $P_n(x)$ is the Legendre polynomial, the substitution $s = n+r$ into (*) gives $(n+r)!P_n^{(-r)} = (n-r)!(x^2-1)^r P_n^{(r)}$, which defines a derivative of P_n of integral negative order $-r$ ($|r| \leqq n$). Recurrence formulas for $D_n^{(s)}$, a determinant expression for $P_n^{(r)}$ and an expansion of $P_n^{(r)}(x)$ into a homogeneous polynomial of $(1+x)^{\frac{1}{2}}$ and $(1-x)^{\frac{1}{2}}$ of degree $2n-2r$ are given. [Note of the reviewer: The function $P_n^{(-r)}(x)$ is, except for the normalization factor, the ultraspherical polynomial $P_{n+r}^{(\lambda)}(x)$, where $\lambda = \frac{1}{2}-r$. Cf. G. Szegö, Orthogonal Polynomials, New York, 1939, equation 4.7.12.] *I. Opatowski* (Minneapolis, Minn.).

Lanzewizky, I. L. Über die Orthogonalität der Féjèr-Szegöschen Polynome. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 199–200 (1941). [MF 4827]

Let α_n be real, $\alpha_n \neq 0$. The system $P_n(x)$ defined by

$$P_n(\cos\theta) = \sum_{k=0}^{[n/2]}{}' \alpha_k \alpha_{n-k} \cos(n-2k)\theta$$

($\sum'$ indicates that the term $k = n/2$, n even, has to be multiplied by $\frac{1}{2}$) has been studied by Fejér and Szegö. The author obtains a necessary and sufficient condition in order that $P_n(x)$ form an orthogonal system in $-\infty, +\infty$ with respect to a suitable distribution $d\psi(x)$. This condition is the following: $c_{n+1}(c_1+c_n-c_2-c_{n-1}) = c_1c_n - c_2c_{n-1}$, $n \geqq 3$; $c_n = \alpha_n/\alpha_{n-1}$. The solution of this difference equation is obtained in terms of simple trigonometric expressions. No proofs are given. *G. Szegö* (Stanford University, Calif.).

Feldheim, E. Sur les polynomes généralisés de Legendre. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 241–254 (1941). (French. Russian translation) [MF 5175]

Given the power series $\sum a_n z^n$ (a_n real). Consider the generalized Legendre polynomials (Fejér), of degree $n = 0, 1, 2, \cdots$, generated by the expansion

$$|F(re^{i\theta})|^2 = \sum P_n(\cos\theta) r^n.$$

Making use of the known explicit expression

$$P_n(\cos\theta) = \sum_{k=0}^{n} a_k a_{n-k} \cos(n-2k)\theta$$

and of the classical recurrence relation for orthogonal polynomials (OP), the author gives conditions under which the sequence $\{P_n(\cos\theta)\}$ is a sequence of OP. The author also studies the distribution of the zeros of $P_n(\cos\theta)$, utilizing known results of Fejér and Szegö. *J. A. Shohat.*

Feldheim, Ervin. On a system of orthogonal polynomials associated with a distribution of Stieltjes type. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 528–533 (1941). [MF 4950]

The author investigates the polynomials

$$l_n^{(\alpha)}(x) = \frac{\Gamma(n+\alpha+1)\, x!}{\Gamma(x+\alpha+1)\, n!} e^{\lambda x} \Delta^n \{e^{-\lambda x}\binom{x+\alpha}{x-n}\}$$

satisfying the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)} e^{-\lambda x} l_n^{(\alpha)}(x) l_m^{(\alpha)}(x) = \frac{\Gamma(n+\alpha+1)}{n!} e^{-n\lambda}(1-e^{-\lambda})^{-\alpha-1}\delta_{nm},$$

$n, m = 0, 1, 2, \cdots$; $\Re\alpha > -1$, $\lambda > 0$. The special case $l_n^{(0)}(x)$ was investigated by M. Gottlieb [Amer. J. Math. **60**, 453–458 (1938)]. A systematical study of the formal properties of the general polynomials $l_n^{(\alpha)}(x)$ is given. They have interesting relations to the Laguerre, Hermite and Jacobi polynomials. *G. Szegö* (Stanford University, Calif.).

Feldheim, Ervin. Quelques résultats sur les polynômes d'Hermite et de Laguerre. Relations avec les fonctions hypergéométriques les plus générales. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 534–537 (1941). [MF 4951]

Let $\phi_n(x_1, \cdots, x_n) = \sum a_{rs} x_r x_s$, $1 \leqq r, s \leqq n$, be a positive definite quadratic form; the author defines the generalized Hermite polynomials $H_{m_1, m_2, \cdots, m_n}(x_1, \cdots, x_n)$ by

$$\exp\{2(h_1\xi_1 + \cdots + h_n\xi_n) - \varphi_n(h_1, \cdots, h_n)\} = \sum \frac{h_1^{m_1} \cdots h_n^{m_n}}{m_1! \cdots m_n!} H_{m_1, \cdots, m_n}(x_1, \cdots, x_n),$$

where $\xi_r = \sum a_{rs} x_s$, $1 \leqq s \leqq n$. The polynomials $H_{m_1, \cdots, m_n}$ associated with the form $a_{rr} = \lambda_r^2 - 1$, $a_{rs} = -1$, $r \neq s$, are expressed in the form of an integral involving the product of n ordinary Hermite polynomials. Various special and limiting cases are considered. *G. Szegö* (Stanford University, Calif.).

Feldheim, Ervin. Équations intégrales pour les polynomes d'Hermite à une et plusieurs variables, pour les polynomes de Laguerre, et pour les fonctions hypergéométriques les plus générales. Ann. Scuola Norm. Super. Pisa (2) **9**, 225–252 (1940). [MF 5464]

Let $\varphi(x) = \varphi(x_1, \cdots, x_n) = \sum a_{ik} x_i x_k$ be a quadratic form of determinant $\Delta \neq 0$, let $\psi(x)$ be the adjoint form, and $\xi_i = \frac{1}{2}\partial\varphi/\partial x_i$. The Hermite polynomials (of the first kind)

in n variables are defined by

$$\exp\{2(h_1\xi_1+\cdots+h_n\xi_n)-\varphi(h_1,\cdots,h_n)\}$$
$$=\sum\frac{h_1^{m_1}\cdots h_n^{m_n}}{m_1!\cdots m_n!}H_{m_1,\cdots,m_n}(x_1,\cdots,x_n)$$

and the adjoint polynomials (of the second kind) by replacing ξ by x, φ by ψ and H by G. The general integral equation reads

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-s^2}H_{m_1}\left(\frac{z+\xi_1}{\lambda_1}\right)\cdots H_{m_n}\left(\frac{z+\xi_n}{\lambda_n}\right)dz$$
$$=\lambda_1^{-m_1}\cdots\lambda_n^{-m_n}H_{m_1,\cdots,m_n}(x_1,\cdots,x_n).$$

An inversion is given in which $H_{m_1,\cdots,m_n}$ occurs under the sign of integration and the product of the H_m's on the right. This formula contains a large number of special cases previously discussed by the author and others. An interesting special case is obtained for $n=2$, $\lambda_1=\lambda_2=1$, $\xi_1=y$, $\xi_2=-x$, the right hand side being $2^m n!y^{m-n}L_n^{(m-n)}(2xy)$, $m\geqq n$. For ultraspherical polynomials the author expresses

$$\int_{-1}^{1}(1-x^2)^{\sigma-\frac{1}{2}}V_n^{(\sigma)}((x+y)/\lambda)dx$$

as a linear combination of $V_{n-2r}^{(\sigma+2r)}(y/\lambda)$, $2r\leqq n$. The formula of Erdélyi

$$\int_0^1 x^{\sigma-1}(1-x)^{\alpha-\sigma-1}F(\alpha,\beta,\gamma,xz)dx=\frac{\Gamma(\sigma)\Gamma(\alpha-\sigma)}{\Gamma(\alpha)}F(\sigma,\beta,\gamma,z)$$

can serve as a prototype for the hypergeometric case. The author derives corresponding integral equations for the general hypergeometric functions F_A and F_D of Lauricella and cites numerous special cases involving Jacobi polynomials, functions of Humbert and of Whittaker in several variables. Laguerre polynomials and Bessel functions are brought in by suitable passages to the limit, and so on. *E. Hille.*

Burchnall, J. L. A note on the polynomials of Hermite. Quart. J. Math., Oxford Ser. 12, 9–11 (1941). [MF 4665]

The formula of Feldheim-Watson [J. London Math. Soc. 13, 22–29 and 29–32, respectively, (1938)] expressing $H_m(x)H_n(x)$ in terms of $H_{m+n-2r}(x)$ is proved in a new way. First the "inverse" formula is proved which expresses $H_{m+n}(x)$ in terms of $H_{m-r}(x)H_{n-r}(x)$. The notation $H_n(x)=e^{x^2}(-D)^ne^{-x^2}$ is used. Finally the following elegant formula is pointed out:

$$H_n(x)=\sum_{r=0}^{n}\binom{n}{r}\bar{H}_r(x)\bar{H}_{n-r}(x),\quad \bar{H}_n(x)=e^{x^2/2}(-D)^ne^{-x^2/2}.$$

G. Szegö (Stanford University, Calif.).

Toscano, Letterio. Relazioni tra i polinomi di Laguerre e di Hermite. Boll. Un. Mat. Ital. (2) 2, 460–466 (1940). [MF 5566]

The author works with the Uspensky and the Kogbetliantz integrals, the former relating $L_n^{(\alpha)}(x^2)$ with $H_{2n}(x)$, the latter expressing $x^\alpha L_n^{(\alpha)}(x)$ as a fractional integral of $x^\beta L_n^{(\beta)}(x)$. He shows that the inversion of the Uspensky integral leads to a special case of the Kogbetliantz integral and that the integrals can be used to obtain linear expressions of $L_n^{(\alpha)}(x^2)$ in terms of $H_{2k}(x)$ and of $L_n^{(\alpha)}(x)$ in terms of $L_k^{(\beta)}(x)$, $k=0,1,\cdots,n$. He gives a new derivation of the expression for $H_n(x)$ as $C_n\lim_{\omega\to\infty}\omega^{-n}L_n^{(\omega^2+k)}(\omega x+\omega^2)$, which he had found earlier, and uses this to express $H_n(x)$ linearly in terms of $L_k^{(\alpha)}(x)$ and to derive the addition theorem for Hermite polynomials from that of Laguerre polynomials.

E. Hille (Palo Alto, Calif.).

Broggi, Ugo. Su qualche espressione integrale dei polinomi di Laguerre. Boll. Un. Mat. Ital. (2) 3, 109–112 (1940). [MF 5580]

Broggi, Ugo. Su di una classe di sviluppi in serie di polinomi di Laguerre. Boll. Un. Mat. Ital. (2) 3, 195–198 (1941). [MF 5585]

In the first note the author remarks that

$$L_n(t)=\frac{1}{n!}\left\{\frac{d^n}{dx^n}[e^{ixt}(x+i)^n]\right\}_{x=0},$$

which gives a representation of $L_n(t)$ by a Cauchy integral. From this he derives a new proof for the fact that $\sum c_nL_n(z)$ converges in the finite complex plane and represents an entire function of exponential type (=function of order one and normal type) when $\limsup_{n\to\infty}|c_n|^{1/n}<1$. In the second note he proves that every such entire function $f(z)$ can be developed in a series $\sum c_nL_n(\alpha z)$, convergent for all z, where $\alpha>2\mu$ and μ is the abscissa of convergence of $\int_0^\infty e^{-st}f(t)dt$. [In particular, it is sufficient that α exceeds twice the type of $f(z)$. Both statements are the best of their kind.] Further, he shows that if $f(x)\sim\sum_0^\infty a_nL_n(x)$, where $\limsup_{n\to\infty}|a_n|^{1/n}=1$ and $z=1$ is a regular point of $\sum a_nz^n$, then the Laguerre series can be "summed" into the form $\sum b_n(\alpha)L_n(\alpha x)$ for every $\alpha>1$, where

$$\alpha^nb_n(\alpha)=\sum_{k=0}^{\infty}\binom{n+k}{k}a_{n+k}((\alpha-1)/\alpha)^k,$$

and the resulting series converges in the whole finite plane and represents an entire function of exponential type. [The author's heuristic argument on p. 112 of the first note is not convincing. His contention on p. 198 of the second note that $\sum(-1)^nL_n(t)$ diverges for all t is erroneous. The Fatou-Riesz theorem applied to the generating function of Laguerre polynomials shows that the series converges for t real, positive.] *E. Hille* (Palo Alto, Calif.).

Banerjee, D. P. On some expansions containing Laguerre polynomials, and their expressions in terms of Whittaker's confluent hypergeometric functions. Philos. Mag. (7) 32, 84–85 (1941). [MF 5091]

Elementary derivations of the known addition theorems

$$L_n^{\alpha+\beta+1}(x+y)=\sum_{s=0}^{n}L_{n-s}^{\alpha}(x)L_s^{\beta}(y)=\sum_{s=0}^{n}((-y)^s/s!)L_{n-s}^{\alpha+\beta+s+1}(x),$$

and their expression in terms of confluent hypergeometric functions. *M. C. Gray* (New York, N. Y.).

Sansone, G. Il teorema di Abel per le serie di polinomi di Jacobi. Boll. Un. Mat. Ital. (2) 3, 1–5 (1940). [MF 5568]

Consider a series of Jacobi polynomials $\sum_{n=0}^{\infty}a_nP_n^{(\alpha,\beta)}(z)$, with $\underline{\lim}_{n\to\infty}|a_n|^{-\frac{1}{n}}=R$, $R>1$, finite. It is known that the region of convergence of this series is the interior of the ellipse with foci at ±1 and sum of the semi-axes equal to R. The author assumes the convergence of the series at a certain point on the circumference of the ellipse and proves for it a theorem analogous to Abel's theorem for power series. Use is made of the known asymptotic expressions $(n\to\infty)$ for $P_n^{(\alpha,\beta)}(z)/P_n^{(\alpha,\beta)}(z_0)$ and $P_{n+1}^{(\alpha,\beta)}(z)/P_n^{(\alpha,\beta)}(z)$.

J. A. Shohat (Philadelphia, Pa.).

Keldych, M. Sur les conditions pour qu'un système de polynômes orthogonaux avec un poids soit fermé. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 778–780 (1941). [MF 4475]

In a simply connected region D, let $h(z)$, $z=x+iy$, be a positive function which is bounded below on any closed subset of D and for which $\iint_D h(z)dxdy<\infty$. A function $f(z)$, analytic in D, is said to belong to the class H_D if

$$\iint_D h(z)\,|f(z)|^2\,dxdy<\infty.$$

The author calls the system of all polynomials $P(z)$ complete in D with respect to the weight function $h(z)$, if, for every $f(z)\in H_D$, the greatest lower bound of

$$\iint_D h(z)\,|f(z)-P(z)|^2\,dxdy$$

is zero. The principal result of the paper is the statement (whose proof is indicated) that for every simply connected region D there exists an $h(z)$ with the above listed properties such that the system of all polynomials is complete in D with respect to $h(z)$. In addition, the author points out that, in general, it is not possible, when given a region D arbitrarily, to have a weight function $h(z)$ which is bounded below uniformly in the whole region D, and for which the system of all polynomials is complete in D. He describes the order in which $h(z)$ may decrease as z approaches the boundary of D so that the desired completeness property will be possessed by the system of all polynomials in D.

S. E. Warschawski (St. Louis, Mo.).

Korovkin, P. P. Sur les polynomes orthogonaux le long d'un contour rectifiable dans le cas de la présence d'un poids. Rec. Math. [Mat. Sbornik] N.S. **9** (51), 469–485 (1941). (Russian. French summary) [MF 5497]

Let C be a simple closed rectifiable contour, $n(\xi)$ non-negative and integrable on C, $\{p_n(x)\}$ the set of polynomials defined by

$$\int_C n(\xi)\overline{p_n(\xi)}p_m(\xi)d\sigma=\delta_{nm},$$

the coefficient of x^n being positive. Let $f(x)$ be regular in the interior of C and having on C limiting values $f(\xi)$ almost everywhere, and such that $\int_C n(\xi)\,|f|^2d\sigma<\infty$. Let $P_n(x)$ be the nth partial sum of the Fourier series expansion of $f(x)$ in the polynomials $\{p_n(x)\}$. It is said that $\{p_n(x)\}$ is closed relative to $f(z)$ if

$$\int_C n(\xi)\,|f(\xi)-P_n(\xi)|^2\,d\sigma\to 0.$$

In extending work of Szegö and Smirnoff the author proves various results among which we mention: (i) the determination of the most general class of functions $f(x)$ with respect to which $\{p_n(x)\}$ is closed, under the assumption of integrability of $|\gamma'(\xi)|\log n(\xi)$, where $z=\gamma(x)$ maps $|z|<1$ conformally into the interior of C; (ii) determination of asymptotic expressions for $p_n(x)$ outside of C under conditions more general than were considered before; also, inside and on C, in case C is analytic. *J. D. Tamarkin.*

Shohat, J. The best polynomial approximation of functions possessing derivatives. Duke Math. J. **8**, 376–385 (1941). [MF 4600]

Let $f(x)$ be continuous on the finite range $a\leqq x\leqq b$ and let $\Pi_n(f;x)$ denote the associated polynomial of degree n of the best approximation. It can be characterized by the following property: There are at least $n+2$ distinct points at which $|f-\Pi_n|=\max|f-\Pi_n|=E_n(f)$, and at which alternately $f-\Pi_n=\pm E_n(f)$ holds. From this property we conclude the existence of $n+1$ points at which $f-\Pi_n=0$. This connection between best approximation and interpolation is the basic tool of the author, in particular the classical representation $f(x)-L_n(x)=f^{(n+1)}(\xi)\Pi(x-c_i)/(n+1)!$, $1\leqq i\leqq n$, where $L_n(x)$ is the interpolation polynomial coinciding with $f(x)$ at c_i (ξ between $\min c_i$ and $\max c_i$). The results are rather close to those of S. Bernstein and de la Vallée Poussin. For instance (a) let $f^{(n+1)}(x)\geqq m>0$, then $E_n(f)\geqq 2^{-2n-1}(b-a)^{n+1}m/(n+1)!$; (b) let $f^{(n+1)}(x)\geqq\varphi^{(n+1)}(x)\geqq 0$, then $E_n(f)\geqq E_n(\varphi)$. Furthermore an interesting "separation theorem" is proved on the $n+2$ pairs of points at which $|f(x)-\Pi_n(f;x)|=E_n(f)$ and $|\varphi(x)-\Pi_n(\varphi;x)|=E_n(\varphi)$, provided neither of the functions $\varphi^{(n)}(x)$, $f^{(n+1)}(x)$, $[E_n(f)]^{-1}f^{(n+1)}(x)\pm[E_{n-1}(\varphi)]^{-1}\varphi^{(n)}(x)$ changes sign in a, b.

G. Szegö (Stanford University, Calif.).

Banach, S. Sur la divergence des interpolations. Studia Math. **9**, 156–165 (1940). (French. Ukrainian summary) [MF 5264]

Let C be the space of continuous functions and $X_i(t)$ be a complete sequence of functions in C satisfying the following condition: If for some $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $0\leqq t_1<t_2<\cdots<t_n\leqq 1$ we have

$$\sum_{i=1}^{n}\alpha_i X_i(t_j)=0, \qquad j=1, 2, \cdots, n,$$

then $\alpha_1=\alpha_2=\cdots=\alpha_n=0$. Denote further the sequence $t_1, t_2, \cdots, t_n$ by T_n. If $f(x)$ is any function in C, there clearly exists a unique sequence of numbers $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_n^{(n)}$ such that

$$\sum_{i=1}^{n}\alpha_i^{(n)}X_i(t_j)=f(t_j).$$

Let

$$U_n(f(x), t)=\sum_{i=1}^{n}\alpha_i^{(n)}X_i(t).$$

If $X_i(t)=t^i$, then $U_n(x, t)$ coincides with the Lagrange interpolation polynomial of $f(x)$ taken at the points T_n. The author proves among others the following theorem: For every $X_i(t)$ we can find an $f(x)$ and a sequence $T_1, T_2, \cdots$ with $T_n\subset T_{n+1}$, such that $\sum_n T_n$ is everywhere dense in $(0, 1)$ and

$$\overline{\lim}\,|U_n(f(x), t)|=+\infty$$

almost everywhere in $(0, 1)$. *P. Erdös.*

Favard, J. Sur l'interpolation. J. Math. Pures Appl. (9) **19**, 281–306 (1940). [MF 4625]

For various classes of functions (for example, the class of functions having bounded derivatives of a given order) the order of the best approximation to a given function by polynomials of a given degree is well known. The author sets himself the problem of determining polynomials furnishing an approximation of the same order, the polynomials to depend linearly on the values taken by the function at a suitable sequence of points. Systematic processes for constructing such interpolating polynomials are given. An interesting auxiliary problem is the determination of the "simplest" interpolating function $g(x)$, with bounded derivative of a given order, taking the same values as a given function $f(x)$ at a finite number of points. If the order is unity, $g(x)$ is the natural piecewise-linear function.

R. P. Boas, Jr. (Durham, N. C.).

Laden, H. N. An application of the classical orthogonal polynomials to the theory of interpolation. Duke Math. J. 8, 591–610 (1941). [MF 5945]

Given an infinite triangular array $\{x_{i,n}\}$ $(i=1, 2, \cdots, n;$ $n=1, 2, \cdots)$ of abscissas belonging to a given interval (a, b) and a corresponding array of real numbers $\{y_{i,n}\}$. Let $f(x)$ be defined on (a, b), with $f(x_{i,n})=y_{i,n}$. The author takes for the $x_{i,n}$ the zeros of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite, and constructs the interpolation polynomial $F_n(x; f)\equiv F_n(x)$, of degree $4n-1$, such that

$$F_n(x_{i,n})=f(x_{i,n});\ F_n^{(\nu)}(x_{i,n})=0,\quad \nu=1, 2, 3;\ i=1, 2, \cdots, n.$$

His object is to discuss the convergence $\lim_{n\to\infty} F_n(x)=f(x)$, $x\varepsilon(a, b)$ or $x\varepsilon(c, d)=$a subinterval of (a, b), for any continuous $f(x)$, properly restricted at infinity, if (a, b) is infinite. [This was first considered by Kryloff and Stayerman for trigonometric abscissas.] Several convergence theorems are established, also an error in Kryloff-Stayerman's proof is corrected. The author uses, for (a, b) finite, the classical method of Fejér, and for (a, b) infinite Shohat's method based on mechanical quadratures. *J. A. Shohat.*

Merli, Luigi. Sulla contemporanea approssimazione di una funzione e della sua derivata con la formula di interpolazione di Lagrange. Boll. Un. Mat. Ital. (2) 3, 116–125 (1940). [MF 5582]

Let $f(x)$ be defined on $(-1, 1)$ and have the first derivative $f'(x)$ which satisfies a Lipschitz condition of order $\alpha>0$. Let $L_n(x)$ be the Lagrange interpolation polynomial of degree not greater than $n-1$, with the interpolation points at $x_k=\cos\theta_k$, $\theta_k=((2k-1)/2n)\pi$, $k=1, 2, \cdots, n$. It is proved that $L_n(x)\to f(x)$, $L_n'(x)\to f'(x)$ as $n\to\infty$, uniformly on $(-1, 1)$. The proof is based on a detailed investigation of $\sum_{k=1}^{n}|l_k'^{(n)}(x)|$, where $L_n(x)=\sum_{k=1}^{n}f(x_k)l_k^{(n)}(x)$.

J. D. Tamarkin (Providence, R. I.).

Curtiss, J. H. Riemann sums and the fundamental polynomials of Lagrange interpolation. Duke Math. J. 8, 525–532 (1941). [MF 5188]

Let C be a rectifiable Jordan curve, M the open interior, K the open exterior of C. Let $z=\varphi(w)$ be the map function of K onto $|w|>1$, $z=\infty\to w=\infty$, $\varphi'(\infty)=c$. Finally let $\omega_n(z)=\Pi_{k=1}^{n}(z-\varphi(\epsilon_k))$, $\epsilon_k=\exp(2\pi ik/n)$. In a classical paper [Nachr. Ges. Wiss. Göttingen 1918] L. Fejér proved that $\lim_{n\to\infty}|\omega_n(z)|^{1/n}$ exists and equals $|c|$ or $|c|\ |w|$ according as $z\varepsilon M$ or $z\varepsilon K$. The present paper gives the following refinement of this result. (1) $c^{-n}\omega_n(z)\to -1$, $z\varepsilon M$; $c^{-n}(w^n-1)^{-1}\omega_n(z)\to 1$, $z\varepsilon K$ (uniformly on every closed set in M and K, respectively). (2) The second part of (1) holds uniformly in $C+K$ provided $\varphi'(w)\neq 0$ and $\varphi'(w)$ is of bounded variation for $|w|=1$. *G. Szegö* (Stanford University, Calif.).

Curtiss, J. H. Necessary conditions in the theory of interpolation in the complex domain. Ann. of Math. (2) 42, 634–646 (1941). [MF 4962]

Let C be a rectifiable Jordan curve, $f(z)$ analytic in the closed interior of C and $L_n(f; z)$ the sequence of Lagrange polynomials associated with $f(z)$ and with interpolation points on C. In his thesis [Budapest, 1926] Kalmár proved that the uniform distribution of the interpolation points in the sense of Fejér is necessary and sufficient for the uniform convergence $L_n(f; z)\to f(z)$. Later Walsh generalized this result replacing C by more general sets. The present investigation gives another approach to these results and also some interesting refinements of the study of the fundamental polynomials $\omega_n(z)$ of the interpolation in question.

G. Szegö (Stanford University, Calif.).

Artmeladze, N. Über Formeln der mechanischen Kubaturen. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 7, 147–160 (1940). (Russian. German summary) [MF 5307]

The author gives formulae for the approximate evaluation of the double integral (1) $\int_{-b}^{a}\int_{-a}^{a}f(x, y)dxdy$, where $f(x, y)$ is continuous with its partial derivatives up to a certain order in the area under consideration, and also for triple integrals. The formulae are analogous to ordinary mechanical quadrature formulae. An approximate estimate of the remainder term is given which enables the author to compare the accuracy of similar formulae given previously by other writers. The procedure consists in subdividing the area into small rectangles and evaluating the corresponding integrals (1) in terms of the values taken at the vertices by the solution $U(x, y)$ of the differential equation $\partial^2U/\partial x\partial y=f(x, y)$. For $U(x, y)$ we use Taylor's formula. [Mention should be made of a recent paper by M. Sadowsky, Amer. Math. Monthly 47, 539–543 (1940); cf. these Rev. 2, 62.]

J. A. Shohat (Philadelphia, Pa.).

Special Functions

Aroian, Leo A. Continued fractions for the incomplete Beta function. Ann. Math. Statistics 12, 218–223 (1941). [MF 4713]

This paper modifies an idea used by Müller [Biometrika 22, 284–297 (1930–1931)] of computing the continued fraction corresponding to the incomplete Beta function $B_x(p, q)$. Müller's continued fraction is in terms of the variable $x/(1-x)$, while that of the present paper is in terms of the variable x. Both continued fractions converge for $-\infty<x<1$. [The range of convergence of Müller's c.f. is incorrectly computed in the present paper.] The argument commencing with the second sentence p. 220 is not convincing. A further investigation of the relationship between the two c.f.'s and of their respective advantages seems to be indicated.

W. Leighton (Houston, Tex.).

Shanker, Hari. On certain integrals and expansions involving Weber's parabolic cylinder functions. J. Indian Math. Soc. (N.S.) 4, 158–166 (1940). [MF 4320]

Starting from the addition theorem that for $\Re(n)\geqq 0$

$$D_n(ax+by)=\Gamma(n+1)e^{\frac{1}{4}(bx-ay)^2}\binom{a}{r}^n\sum_{m=0}^{\infty}\frac{D_{n-m}(rx)D_m(ry)}{\Gamma(n-m+1)m!}\binom{b}{a}^m$$

and for $\Re(n)<0$

$$D_n(ax+by)=\frac{1}{\Gamma(-n)}e^{\frac{1}{4}(bx-ay)^2}\binom{a}{r}^n\sum_{m=0}^{\infty}\frac{\Gamma(m-n)D_{n-m}(rx)D_m(ry)}{m!}\binom{-b}{a}^m$$

where $r^2=a^2+b^2$, a and b are numbers, either real or imaginary, such that a/r is real, the author derives series expansions and integral formulas involving one or two parabolic cylinder functions. Not all of the resulting formulas are new, but they have mostly appeared previously in widely scattered journals. *M. C. Gray* (New York, N. Y.).

Brauer, Peter und Brauer, Elfriede. Über unvollständige Anger-Webersche Funktionen. Z. Angew. Math. Mech. 21, 177–182 (1941).

The authors consider the functions

$$u=u_p^{\tau}(\sigma)=\int_0^{\sigma}\cos(\tau\sin\sigma-p\sigma)d\sigma,$$

$$v=v_p^{\tau}(\sigma)=\int_0^{\sigma}\sin(\tau\sin\sigma-p\sigma)d\sigma,$$

which, for $\sigma=\pi$, are related to the Bessel and Weber functions through the expressions $u_p^{\tau}(\pi)=\pi J_p(\tau)$, $v_p^{\tau}(\pi)=-\pi W_p(\tau)$. Using Jacobi's results for the Fourier series expansions of $\cos(\tau\sin\sigma)$ and $\sin(\tau\sin\sigma)$ (which involve the Bessel function of integral order as coefficients), the authors obtain series representations for the functions u and v. For the particular case $p=\tau$, tables for u/π and $-v/\pi$ are computed. These give the values of these functions to six decimals for values of σ/π (σ=upper limit) ranging from 0 to 1.00 and for values of p from 0 to 0.50, at intervals of 0.05 for each. *M. A. Basoco* (Lincoln, Nebr.).

Erdélyi, A. Generating functions of certain continuous orthogonal systems. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 61–70 (1941). [MF 5163]

The integral representation of the parabolic cylinder functions $D_n(z)$, $\Re n<0$, in combination with Mellin's formula, furnishes the "generating function"

$$2\pi i\exp(-\tfrac{1}{4}z^2-zt-\tfrac{1}{2}t^2)=\int_{c-i\infty}^{c+i\infty}t^n\Gamma(-n)D_n(z)dn,$$

$c<0$, $|\arg t|<\pi/4$. Using the calculus of residues, this furnishes the ordinary generating function of $D_n(z)$, n integer. A similar argument leads to certain other generating functions involving $D_n(x)D_{-n-1}(iy)+D_n(-x)D_{-n-1}(-iy)$. Corresponding considerations are possible for confluent hypergeometric functions, for conal harmonics, as well as for ordinary hypergeometric functions. *G. Szegö.*

Kesava Menon, P. A generalization of Legendre functions. J. Indian Math. Soc. (N.S.) 5, 92–102 (1941). [MF 5699]

The generalization under consideration is

$$P_{n,s}(z)=\frac{1}{n!s^n}\frac{d^n}{dz^n}(z^s-1)^n;$$

it is shown to have properties similar to those of the Legendre polynomials. *O. Szász* (Cincinnati, Ohio).

Somigliana, Carlo. Funzioni trascendenti del campo gravitazionale ellissoidico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 597–611 (1940). [MF 4790]

The author derives certain relations which are satisfied by transcendental functions Φ_a, Φ_b, Φ_c which occur in the theory of the gravitational field due to an ellipsoid rotating with angular velocity "ω" about its minor axis. These relations facilitate the series development of the functions in question. These functions are defined by equations of the form

$$\omega^2\Phi_a(e'^2,e^2)=g_b/b-g_c/c,$$

where a,b,c $(a>b>c)$ are the semi-axes of the ellipsoid, $e'^2=(a^2-b^2)/a^2$, $e^2=(a^2-c^2)/a^2$, and g_a, g_b, g_c are the values of the gravitational accelerations at the extremities of the respective axes. Using results obtained in a former paper [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 69, 345–357 (1934)] the author finds three functional equations which are satisfied by the Φ's. These equations are of the form:

$$\Phi_a\left(\frac{-x}{1-x},\frac{x-y}{1-x}\right)+\Phi_b(x,y)=0.$$

Since $\Phi_a+\Phi_b+\Phi_c=0$, these functional relations reduce the calculation of the three functions to that of a single one, say Φ_a. From these results, the author obtains, after rather lengthy computations, the series developments of the functions. A typical result is:

$$\Phi_a(x,y)=-\frac{5}{2}+x-\frac{9}{14}y+\frac{38}{147}x^2+\frac{85}{294}xy-\frac{47}{98}y^2+\cdots,$$

$$\Phi_a\left(\frac{x-y}{x+1},\frac{x}{x+1}\right)=-\frac{5}{2}-y+\frac{5}{14}y^2+\frac{19}{98}xy-\frac{85}{294}x^2+\cdots.$$

All expansions are carried through terms of the third order. No discussion is given concerning the domain of validity of these expansions. *M. A. Basoco* (Lincoln, Nebr.).

Chu, L. J. and Stratton, J. A. Elliptic and spheroidal wave functions. J. Math. Phys. Mass. Inst. Tech. 20, 259–309 (1941). [MF 5069]

The differential equation

$$(1-z^2)U''-2(a+1)zU'+(b-c^2z^2)U=0,\quad b=l(l+2a+1)+\epsilon_l,$$

is satisfied for $z\gg1$ by a series of cylinder functions

$$U(a,c;z)=(cz)^{-s-\frac{1}{2}}\sum_n a_nZ_{n+s}(cz),$$

where $s=\rho+a+\frac{1}{2}$ and $0\leqq\rho<1$. The coefficients a_n are given by a three term recurrence relation which is independent of the particular type of cylinder function $Z_n(x)$ that is used. The suffix n takes all odd or even integral values but with a special choice of ρ the negative integral values of n may be eliminated. Another type of solution is obtained by expanding $(z^2-1)^a(cz)^{\frac{1}{2}-s}U$ in a series of cylinder functions, and a set of eight spheroidal functions is listed in which different types of cylinder functions are used in the summation. For the domain $|z|<1$ a set of eight solutions is obtained by using expansions in series of metaspherical functions which the authors call Gegenbauer functions. The formulae relating to these functions are listed, the notation being different from that adopted by N. Nielsen.

The method of continued fractions is used to calculate the coefficients a_n in the various series and appropriate values of the separation constant b. The case $a=-\frac{1}{2}$, which leads to the Mathieu functions, is considered in some detail and the second solution of Mathieu's equation is discussed. The different types of spheroidal functions and Mathieu functions are finally specified in a handy form and some remarks are made on the present status of the work of computation. The separation constants for both prolate and oblate spheroidal functions have been computed for $c=0(.2)5$ and $m,l=0(1)3$, $m+l\leqq3$. The coefficients in the expansions have been calculated for the same values. For the Mathieu functions the ranges $c=0(.2)45$, $l=1(1)4$ are used for the separation constants b_l and the coefficients in the expansions. *H. Bateman* (Pasadena, Calif.).

Rutgers, J. G. Sur des séries et des intégrales définies contenantes les fonctions de Bessel. I, II. Nederl. Akad. Wetensch., Proc. 44, 464–474, 636–647 (1941). [MF 5024, 4983]

The first two parts of this paper comprise a total of 163 formulas, which may be regarded either as expressing series

of Bessel functions in the form of definite integrals or as evaluating definite integrals in terms of Bessel functions. From the latter point of view the basic integrands (all integrated with respect to α between the limits zero and x) are

$$\frac{J_\nu(\alpha)}{\alpha}\sin(x-\alpha),\quad J_\nu(x-\alpha)\frac{\sin\alpha}{\alpha},\quad \frac{J_\nu(\alpha)}{\alpha}\cos(x-\alpha),$$

$$J_\nu(x-\alpha)J_\rho(\alpha)\frac{x-\alpha}{\alpha},\quad J_\nu(x-\alpha)\alpha^\rho,\quad \frac{J_\nu(\alpha)}{\alpha}(x-\alpha)^\rho,$$

$$J_\mu(\alpha)J_\nu(x-\alpha)\alpha^\rho,\quad J_\nu(x-\alpha)J_\rho(\alpha).$$

These are evaluated as infinite series of Bessel functions and the remainder of the paper consists in giving special values to the parameters μ, ν and ρ and constructing linear combinations of pairs of integrals which lead to integrals expressible in closed form. Typical examples of the formulas obtained are

$$\int_0^x J_1(\alpha)\cos\alpha d\alpha/\alpha=\sin xJ_0(x)-\cos xJ_1(x),$$

$$\pi^{-\frac{1}{2}}\int_0^x \cos(\pi/4-\alpha)\alpha^{-\frac{1}{2}}d\alpha=\sum_{n=0}^{\infty}J_{n+\frac{1}{2}}(x).$$

The reader should note that the formulas are not free from numerical errors, though these are usually easily recognized, and that the author has used the notation $I_\nu(x)$ throughout to denote the Bessel function of the first kind generally written $J_\nu(x)$. *M. C. Gray* (New York, N. Y.).

Rutgers, J. G. Sur des séries et des intégrales définies contenantes les fonctions de Bessel. III, IV. Nederl. Akad. Wetensch., Proc. **44**, 744–753, 840–851 (1941). [MF 5121, 5921]

An additional 142 formulas [see the preceding review]. *M. C. Gray* (New York, N. Y.).

Shastri, N. A. Some relations between Bessel functions of third order and confluent hypergeometric functions. Proc. Indian Acad. Sci., Sect. A. **13**, 521–525 (1941). [MF 5245]

A typical formula is

$$(x^{\frac{1}{2}}/y)^{\frac{1}{2}(m-2n)}J_{m,n}[3(xy)^{\frac{1}{2}}]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{-\alpha-1}{n}}{\Gamma(\alpha+n+1)}x^{\frac{1}{2}n}y^{\frac{1}{2}(\alpha-1)}e^{\frac{1}{2}y}J_{n+m}(2x^{\frac{1}{2}})M_{n+(\alpha+1)/2,\,\alpha/2}(y),$$

where the "Bessel function of the third order" is defined by the series

$$J_{m,\alpha}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{3}x)^{m+\alpha+3n}}{n!\Gamma(m+n+1)\Gamma(\alpha+n+1)}.$$

M. C. Gray (New York, N. Y.).

Varma, R. S. On Humbert functions. Ann. of Math. (2) **42**, 429–436 (1941). [MF 4292]

The author's definition of Humbert functions is

$$J_{m,n}(x)=\frac{(x/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}{}_0F_3(m+1,n+1;-x^2/27).$$

The present paper deals with special series of the types $\sum_{n=1}^{\infty}A_nJ_{m,n}(x)$ and $\sum_{n=1}^{\infty}A_nJ_{ln+k,n}(x)$. Also formulas involving an operational symbol are derived which connect $J_{m,n}(x)$ with Whittaker's function. Finally formulas (without operational symbols) are obtained involving $J_{m,n}(x)$ and Hermite's functions, and furthermore $J_{m,n}(x)$ and Neumann's polynomials. *G. Szegö* (Stanford University, Calif.).

***Bateman, H. Some definite integrals occurring in aerodynamics.** Theodore von Kármán Anniversary Volume, pp. 1–7. California Institute of Technology, Pasadena, Calif., 1941.

The integrals

$$\int_0^{\pi/2}(1-\sin\theta)^\rho\sec^{\nu+1}\theta\sin(r\theta)d\theta$$

are expressed in terms of the generalized hypergeometric function. Four integrals are evaluated involving Legendre functions of the second kind. *G. Szegö.*

Daum, John A. On certain basic series. Bull. Amer. Math. Soc. **47**, 781–784 (1941). [MF 5488]

The author discusses the possibility of obtaining new results similar to the relation

$$\sum_{n=0}^{\infty}\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-z)(1-qz)\cdots(1-q^nz)}\frac{z^{n+1}}{1-q^{n+1}}=\sum_{n=0}^{\infty}\frac{q^nz}{(1-q^nz)^2}$$

given by Bailey [J. London Math. Soc. **11**, 156–160 (1936)], by using the complete set of basic series analogues of the Thomae two-term transformations involving the hypergeometric series ${}_3F_2$. He finds that in the entire set of identities (in either the ${}_3F_2$ or basic series) only two are essentially distinct and concludes that further numerical relations will only be obtained by specialization of the basic series parameters. *M. C. Gray* (New York, N. Y.).

Erdélyi, A. Integration of the differential equations of Appell's function F_4. Quart. J. Math., Oxford Ser. **12**, 68–77 (1941). [MF 5218]

Appell's fourth hypergeometric function of two variables

$$F_4(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!(\gamma)_m(\gamma')_n}x^my^n$$

satisfies a system of two partial differential equations of the second order:

$$(*)\qquad \begin{aligned} L_1(Z)&=x(1-x)r-y^2t-2xys+\{\gamma-(\alpha+\beta+1)x\}p\\ &\quad-(\alpha+\beta+1)yq-\alpha\beta Z=0,\\ L_2(Z)&=y(1-y)t-x^2r-2xys+\{\gamma'-(\alpha+\beta+1)y\}q\\ &\quad-(\alpha+\beta+1)xp-\alpha\beta Z=0.\end{aligned}$$

The author gives a general solution of the system (*) in the form

$$Z=\int_C t^{-\gamma}(1-t)^{-\gamma'}f\left(\frac{x}{t}+\frac{y}{1-t}\right)dt,$$

where C is any closed contour in the t-plane and $f(\omega)$ is any branch of the Riemann P-function

$$P\begin{pmatrix}0 & 1 & \infty & \\ 0 & 0 & \alpha & \omega\\ 2-\gamma-\gamma' & \gamma+\gamma'-\alpha-\beta-1 & \beta & \end{pmatrix}.$$

Introducing new variables $x=XY$, $y=(1-X)(1-Y)$ which rationalize the singularities of the P-function, a double

integral form for the general integral of (*) is

$$Z=\int\int \sigma^{\beta-\gamma}(\sigma-1)^{\beta-\gamma'}(\sigma-X)^{1-\gamma-\gamma'+\alpha}\tau^{\alpha-\gamma}(\tau+1)^{\alpha-\gamma'} \times(\tau+Y)^{-\alpha}\{(\sigma-X)\tau-X(1-\sigma)\}^{\gamma+\gamma'-\alpha-\beta-1}d\sigma d\tau,$$

with suitable contours of integration in the σ, τ planes. The author discusses the fundamental systems of solutions, various transformations connecting different fundamental systems and also the monodromic group of (*).

M. C. Gray (New York, N. Y.).

Bailey, W. N. On the double-integral representation of Appell's function F_4. Quart. J. Math., Oxford Ser. **12**, 12–14 (1941). [MF 4666]

A recent formula of Burchnall-Chaundy [Quart. J. Math., Oxford Ser. **11**, 249–270 (1940); cf. these Rev. **2**, 287] expressing F_4 in the form of an infinite series involving F is proved in a new manner. A double-integral representation for $(1-x)^{-a}(1-y)^{-b}F_4[a, b; c, c'; -x(1-x)^{-1}(1-y)^{-1}, -y(1-x)^{-1}(1-y)^{-1}]$ is also derived. *G. Szegö.*

Burchnall, J. L. and Chaundy, T. W. Expansions of Appell's double hypergeometric functions. II. Quart. J. Math., Oxford Ser. **12**, 112–128 (1941). [MF 5223]

The expansions of the first part [Quart. J. Math., Oxford Ser. **11**, 249–270 (1940); these Rev. **2**, 287] are extended and various interesting special cases are considered. The confluent case is discussed in detail. Finally integral representations are obtained. *G. Szegö.*

Meijer, C. S. Integraldarstellungen für Whittakersche Funktionen und ihre Produkte. I. Nederl. Akad. Wetensch., Proc. **44**, 435–441 (1941). [MF 5021]

The author first recalls some properties of his G-function and the expressions for numerous other functions by means of it. He gives in particular an expression for $M_{k,m}(z)$, three expressions for $W_{k,m}(z)$ and expressions for $W_{k,m}(z)W_{-k,m}(z)$, $W_{k,m}(iz)W_{k,m}(-iz)$. He finally obtains an equation

$$\Gamma(2\lambda)W_{k,m}(z) =z^{\frac{1}{2}}e^{\frac{1}{2}z}\int_1^\infty e^{-\frac{1}{2}zt}W_{k+\lambda,\mu}(zt)F(a, b; 2\lambda; 1-t)(t-1)^{2\lambda-1}t^{\mu-\frac{1}{2}}dt,$$

where $a=\lambda+\mu-m$, $b=\lambda+\mu+m$, $\Re(\lambda)>0$, $z\neq 0$, $|\arg z|<\frac{1}{2}\pi$. This equation includes as particular cases some equations previously obtained [cf. the following review].

H. Bateman (Pasadena, Calif.).

Meijer, C. S. Integraldarstellungen für Whittakersche Funktionen und ihre Produkte. II. Nederl. Akad. Wetensch., Proc. **44**, 599–605 (1941). [MF 4978]

[Cf. the preceding review.] The author first obtains the relation

$$\pi^{\frac{1}{2}}W_{k,m}(2x^2)=2^{\frac{1}{2}-k}x\int_0^\infty e^{-x^2\operatorname{sh}^2 t}D_{2k}(2x\operatorname{ch} t)\operatorname{ch} 2mt\,dt,$$

which is a particular case of a more general one involving the associated Legendre function, from which some known relations are derived. Some other integral expressions for $W_{k,m}(z)$ in terms of associated Legendre functions are found and a very simple equation

$$\pi^{\frac{1}{2}}D_n(z)=2^{\frac{1}{2}n+1}e^{-\frac{1}{4}z^2}\int_0^{\frac{1}{2}\pi}e^{-\frac{1}{2}z^2\tan^2\phi}\sec^{n+2}\phi\cos n\phi\,d\phi$$

is obtained for the parabolic cylinder function $D_n(z)$. By some developments involving his G-function, the author obtains the equation

$$\pi^{\frac{1}{2}}W_{k,m}(z)=(2z^2e^{\pi i})^{\frac{1}{4}}\int_0^\infty K_m(z\operatorname{ch} t)P_{m-\frac{1}{2}}^{k}(\operatorname{ch} t)\operatorname{sh}^{1-k}t\operatorname{ch}^{k+\frac{1}{2}}t\,dt, \quad z\neq 0;\ |\arg z|<\tfrac{1}{2}\pi;\ \Re(k)<1,$$

and two very similar equations obtained previously in another way. Integrals for the product $W_{k,m}(iz)W_{k,m}(-iz)$ are obtained which involve either a Lommel function $S_{2k,2m}(z\sec\phi)$ or $S_{2k+m-\frac{1}{2},m+\frac{1}{2}}(z\sec\phi)$. *H. Bateman.*

Meijer, C. S. Neue Integraldarstellungen für Whittakersche Funktionen. II. Nederl. Akad. Wetensch., Proc. **44**, 186–194 (1941). [MF 4394]

[For the first communication, cf. these Rev. **2**, 287; for the continuation, cf. the following reviews.] Seven different integral representations of Whittaker's function $W_{k,m}(z)$ are given in which the integrand is a multiple of one of Meijer's G-functions with argument zt and with six parameters of which three depend on k and m. The multiplier is a function of t only and consists of three factors, one of which is a confluent hypergeometric function of either the $W_{k,m}$ or $M_{k,m}$ type. There is also an exponential factor and a power of t. Three similar integral representations are given for $M_{k,m}(z)$. The paper contains a long list of Bessel functions, Struve functions, Lommel functions and confluent hypergeometric functions which can be expressed in terms of the G-function and representations are given also for various products of such functions, the argument of the G-function in each case being either z^2 or $\frac{1}{4}z^2$, where z is the argument of the function represented. *H. Bateman.*

Meijer, C. S. Neue Integraldarstellungen für Whittakersche Funktionen. III. Nederl. Akad. Wetensch., Proc. **44**, 298–307 (1941). [MF 5012]

[Cf. the preceding review.] Integral representations of $W_{k,m}(z^2)$ are obtained by means of contour integrals in which one factor is the Lommel function $S_{2k,2m}(2zu)$. The first representation is subject to the restrictions $z\neq 0$, $|\arg z|<\frac{3}{4}\pi$, while the second one holds for all values of $\arg z$. Some related integral expressions are given in which Struve functions and various kinds of Bessel functions occur.

H. Bateman (Pasadena, Calif.).

Meijer, C. S. Neue Integraldarstellungen für Whittakersche Funktionen. IV. Nederl. Akad. Wetensch., Proc. **44**, 442–451 (1941). [MF 5022]

[Cf. the preceding reviews.] The author obtains two new integral representations which are included in

$$W_{k,m}(z^2)\Gamma(l+m)\Gamma(l-m) =4\pi^{\frac{1}{2}}z^{1-s}e^{-\frac{1}{2}z^2}\int_0^\infty I_s(zu)K_{2m}(zu)W_{l,m}(u^2)e^{-\frac{1}{2}u^2}u^s\,du,$$

where $l=\frac{1}{2}-k$, $z\neq 0$. The number s may be $-2k$ when $\Re(l\pm m)>0$ and may be $-2k-1$ when $\Re(k)\pm\Re(m)<0$. Some special cases are considered and many other integrals involving products of Bessel functions of various kinds are evaluated. *H. Bateman* (Pasadena, Calif.).

Meijer, C. S. Neue Integraldarstellungen für Whittakersche Funktionen. V. Nederl. Akad. Wetensch., Proc. **44**, 590–598 (1941). [MF 4977]

[Cf. the preceding reviews.] Contour integrals for $W_{k,m}(z^2)$ are found which involve products of various types of Bessel

functions of argument zu. The other factor is a confluent hypergeometric function of some type with parameter u^2. Some contour integrals for $W_{k,m}(z^2)$ and $M_{k,m}(z^2)$ have integrands involving products of W or M functions of arguments $2izu$, $-2izu$, respectively. *H. Bateman.*

Thiruvenkatachar, V. R. Operational methods and Neumann series. J. Mysore Univ. Sect. B. 1, 123–132 (1941). [MF 5139]

The operational equivalent of a Neumann expansion

$$(1) \qquad f(z)=\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$$

is an expansion of $\phi(p)$, the operational form of $f(z)$, which can be written

$$(2) \qquad \frac{1+s^2}{1-s^2}\phi\left(\frac{1-s^2}{2s}\right)=\sum_{n=1}^{\infty} a_n s^{\nu+n},$$

where $s=(p^2+1)^{\frac{1}{2}}-p$. When $f(z)$ has an operational form for which the expansion (2) can be written, the coefficients a_n in (1) are thus formally determined. The present paper gives further examples of this method of obtaining Neumann expansions. In particular, new expansions for Struve's functions and new forms for the sum of the Neumann series summed by Kapteyn are obtained. *R. V. Churchill.*

Differential Equations

Roth, L. On the solution of certain differential equations of the second order. Philos. Mag. (7) 32, 155–164 (1941). [MF 5319]

The author considers the differential equation

$$(1) \qquad Ay''+By'^2+Cy'+D=0$$

for cases where A, B, C, D are polynomials of various degrees in y. For the case where A, B, C are of the first degree and D is of the second, he reduces (1) to the form of two simultaneous first order equations involving a further function μ and obtains for μ a second order equation which he solves in various special cases. Cases admitting a first integral linear equation are investigated as well as cases where in (1) A, B, C are of the second degree and D of the third degree in y. *H. Poritsky* (Schenectady, N. Y.).

Rainville, E. D. The factorization of certain second order polynomial differential operators. Amer. Math. Monthly 48, 519–521 (1941). [MF 5714]

The author is concerned with the actual factorization of the operator

$$(*) \qquad a_2D^2+a_1D+a_0, \qquad D\equiv d/dx,$$

where a_2, a_1 and a_0 are polynomials in x, into factors whose coefficients are again polynomials. Using results obtained in a former paper on polynomial solutions of Riccati equations [Amer. Math. Monthly 43, 473–476 (1936)], he proves in case $a_2\equiv 1$: the operator (*) has no proper divisor if the degree of $\Delta=a_1^2-4a_0+2a_1'$ is odd; it has no proper divisors other than $D+\frac{1}{2}a_1\pm\frac{1}{2}[\sqrt{\Delta}]$ if the degree of Δ is even. ($[\sqrt{\Delta}]$ is defined as the polynomial part of the expansion of $\sqrt{\Delta}$ in descending powers of x.) The paper is concluded with remarks concerning the general case ($a_2\not\equiv 1$), which is again reduced to the determination of polynomial solutions of a Riccati equation which, however, is of a less simple type than in the case $a_2\equiv 1$. *E. Rothe* (Oskaloosa, Iowa).

Calamai, Giulio. Sulle soluzioni della equazione caratteristica relativa alla equazione differenziale lineare, omogenea, del secondo ordine, a coefficienti periodici. Boll. Un. Mat. Ital. (2) 3, 370–372 (1941). [MF 5704]

Estimates are given for the two roots of the characteristic equation, corresponding to a differential equation of second order whose coefficients are periodic. *O. Szász.*

Caligo, D. Comportamento asintotico degli integrali dell'equazione $y''(x)+A(x)y(x)=0$, nell'ipotesi $\lim_{x\to+\infty} A(x)=0$. Boll. Un. Mat. Ital. (2) 3, 286–295 (1941). [MF 5599]

The author proves that, if $A(x)$ is real-valued, continuous and $A(x)=O(x^{-(2+\rho)})$, $\rho>0$, as $x\to+\infty$, then each solution of the second order differential equation $y''+A(x)y=0$ has the property that $\lim_{x\to+\infty}[y(x)/x]=\lim_{x\to+\infty} y'(x)$ exists and is finite. The proof given depends in turn upon a result of Sansone for an equation of this sort for which $A(x)=O(x^{-(2+\rho)})$, $\rho>0$, as $x\to+\infty$. *W. T. Reid.*

Franzoni, Triestina. Sull'integrazione approssimata di una particolare equazione differenziale. Boll. Un. Mat. Ital. (2) 3, 32–36 (1940). [MF 5574]

The author shows how to find an approximate integral y_1 of the differential equation $y''+\lambda y=0$, where λ is a function of x. The problem of finding the interval $0\leqq x\leqq\delta$ in which the magnitude of the error $|y-y_1|$ is less than a preassigned $\epsilon>0$ is not completely solved in the paper.
F. G. Dressel (Durham, N. C.).

Mambriani, A. La derivazione d'ordine qualunque e la risoluzione dell'equazione ipergeometrica. Boll. Un. Mat. Ital. (2) 3, 9–18 (1940). [MF 5570]

Using the well-known extension of the differential operator of order ω (ω real or complex)

$$(1) \qquad D^{\omega}f(x)\equiv D_{x_0}^{\omega}f(x)\equiv(\Gamma(-\omega))^{-1}\int_{x_0}^{x}(x-t)^{-\omega-1}f(t)dt,$$

the author shows that the classical hypergeometric equation

$$(2) \qquad x(1-x)y''-\{(\alpha+\beta+1)x-\gamma\}y'-\alpha\beta y=0$$

may be generalized in the form

$$(3) \qquad D^{\alpha}x^{\alpha-\gamma+1}(1-x)^{\gamma-\beta}Dx^{\gamma-\alpha}(1-x)^{\beta-\gamma+1}D^{1-\alpha}y=0.$$

It is then shown that (3) has the solution $y=c_1y_1+y_2$, where c_1 is an arbitrary constant and

$$y_1\equiv D^{\alpha-1}\{x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}\},$$

$$y_2\equiv D^{\alpha-1}\{x^{\alpha-\gamma}(1-x)^{\gamma-\beta-1}\int x^{\gamma-\alpha-1}(1-x)^{\beta-\gamma}D^{-\alpha}0dx\},$$

where $D^{-\alpha}0$ is to be suitably determined. For the purposes of this paper, the author limits himself to the particular determination $D^{-\alpha}0\equiv x^{\alpha-1}/\Gamma(\alpha)$. It is also shown that y_1 contains, as special cases, the definite integral solutions of (2) due to Euler and Jacobi [see Goursat, Actual. Sci. Ind., no. 333, Paris, 1936, p. 29]. Likewise it is shown that y_2 (with $x_0=2$ in (1)) reduces to $(\gamma-1)^{-1}F(\alpha,\beta,\gamma,x)$.

This paper may be regarded as a continuation of the author's paper in Ann. Scuola Norm. Super. Pisa (2) 9, 27–43 (1940) [these Rev. 2, 48]. *M. A. Basoco.*

Mambriani, A. Risoluzione di una classe d'equazioni differenziali lineari. Boll. Un. Mat. Ital. (2) 3, 198–201 (1941). [MF 5586]

L'auteur considère l'équation

$$P_n y^{(n)}+\omega P_n' y^{n-1}+\binom{n}{1}P_n'' y^{n-2}+\cdots+\binom{n}{n}P_n^{(n)}y=0,$$

où ω est une constante et $P_n=P_n(x)$ un polynome de degré n. En applicant une formule de dérivation généralisée donnée dans une note antérieure [cf. l'article précédent], il forme les integrales générales de l'équation, dans les différents cas que les racines de P_n soient simples ou multiples et que ω soit quelconque ou soit un entier positif moins que n.

B. Levi (Rosario).

Erougin, N. Une remarque sur l'article de L. Shifner. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 377–380 (1941). (Russian. French summary) [MF 5842]

The note considers the system

$$dY/dx=Y[U_1\cdot\phi_1(x)+U_2\cdot\phi_2(x)],$$

when U_1 and U_2 are constant square matrices, ϕ_1 and ϕ_2 are scalars and Y is a square matrix of the same order as U_1 and U_2. L. Shifner showed under what conditions the solution is obtainable in "finite form," that is, expressible in terms of a finite number of successive commutators of U_1 and U_2 [Bull. Acad. Sci. URSS. Sér. Math. 4, 341–348 (1940); cf. these Rev. 2, 197]. The present note solves the problem under milder conditions. If $Y=e^S e^{U_2 L_2}$, where $L_2=\int_0^x\phi_2(x)dx$, then $dS/dx=e^{U_2 L_2}\cdot U_1\cdot e^{-U_2 L_2}\phi_1$, and to get a finite solution $(S, dS/dx)$ must vanish. It is shown that for this it is necessary and sufficient that $(U_1, (U_2, U_1))=0$ and that the elementary divisors of U_1 be simple.

M. S. Knebelman (Pullman, Wash.).

Marrack, P. E. Notes on the two-centre problem in wave mechanics. II. Some new solutions of the wave equation. Proc. Cambridge Philos. Soc. 37, 384–396 (1941). [MF 5546]

Solutions of the two differential equations occurring in the two-centre problem are expressed as series of confluent hypergeometric functions. Substitution in the differential equations leads to a 3-term recurrence relation for the determination of the unknown coefficients, and to equations for the eigenvalues which involve infinite continued fractions. The solutions thus obtained are more rapidly convergent for the case of small internuclear distance than those previously found. The convergence of the series used is fully discussed. *A. F. Stevenson* (Toronto, Ont.).

Conforto, Fabio. Sull'integrazione di un sistema di equazioni, relativo alla teoria dello strato limite gassoso. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 2, 127–137 (1941). [MF 5764]

Study of the following system of ordinary differential equations:

$$yy''+xz^\alpha=0, \qquad 0\leqq x\leqq 1,$$
$$(1-\sigma)y'z'+(z''+\sigma a)y=0,$$

with the boundary conditions $y'(0)=0$, $z(0)=b$, $y(1)=0$, $z(1)=1$. Here α, σ, a, b are given constants. *W. Prager.*

García, Godofredo. Reduction of the oscillation equations of a projectile to Volterra's equation of second order. Actas Acad. Ci. Lima 3, 84–90 (1940). (Spanish) [MF 3267]

The solution of the system of equations

$$(1)\quad dx/d\theta=\theta_1 y-\theta_2 z,\quad dy/d\theta=-\theta_1 x+z,\quad dz/d\theta=\theta_2 x-y$$

is reduced to the form

$$(2)\quad x(\theta)=f(\theta)+\int_{\theta_0}^{\theta}x(s)N(\theta,s)ds,$$

$$(3)\quad y(\theta)=\phi(\theta)-\int_{\theta_0}^{\theta}x(s)[\theta_1(s)\cos(\theta-s)-\theta_2(s)\sin(\theta-s)]ds,$$

$$(4)\quad z(\theta)=\psi(\theta)+\int_{\theta_0}^{\theta}x(s)[\theta_1(s)\sin(\theta-s)+\theta_2(s)\cos(\theta-s)]ds,$$

where $f(\theta)$, $\phi(\theta)$, $\psi(\theta)$ are known functions, and the nucleus

$$N(\theta,s)=-\theta_1(s)\int_{\theta_0}^{\theta}[\theta_1\cos(\theta-s)-\theta_2\sin(\theta-s)]d\theta$$
$$+\theta_2(s)\int_{\theta_0}^{\theta}[\theta_1\sin(\theta-s)+\theta_2\cos(\theta-s)]d\theta.$$

Equation (2) is an integral equation of the second kind [Volterra] with $\lambda=1$. K. Popoff had previously obtained a similar result for the special case in which $\theta_1=\theta_2=0$.

E. J. Moulton (Evanston, Ill.).

Sona, Luigi. Osservazione sulle soluzioni statiche di un sistema differenziale del primo ordine. Boll. Un. Mat. Ital. (2) 3, 218–222 (1941). [MF 5592]

The following obvious theorem is proved: If a system $dx_i/dt=X_i(x_1,\cdots,x_n,t)$ admits an integral $\varphi(x_1,\cdots,x_n)=$const., not containing t explicitly, and if φ is stationary at the point (x^0) with a non-vanishing Hessian determinant, then (x^0) is an equilibrium point of the system, that is, $X_i(x_1^0,\cdots,x_n^0,t)=0$, $i=1,\cdots,n$. This follows immediately by differentiation of the obvious identity $\sum_{j=1}^n(\partial\varphi/\partial x_j)X_j\equiv 0$ (in $x_1,\cdots,x_n,t$) with respect to x_i and then setting $(x)=(x^0)$. The author somewhat complicates the proof by using Taylor's development. The connection with Dirichlet's criterion of stability, as generalized by Levi-Civita, is discussed.

D. C. Lewis (Durham, N. H.).

De Simoni, Franco. Su un metodo di integrazione di un sistema di equazioni differenziali del 1° ordine dipendenti da un parametro. Boll. Un. Mat. Ital. (2) 3, 300–307 (1941). [MF 5601]

For a differential system of the form

$$dx_i/dt=f_i(x_1,x_2,\cdots,x_n,t,\rho),\quad i=1,2,\cdots,n,$$

that involves analytically the parameter ρ the author considers the question of embedding a particular solution $x_i=x_{i,0}(t)$ for $\rho=0$ in a family of solutions $x_i=x_i(t,\rho)$, where the functions $x_i(t,\rho)$ are power series in ρ. The author was evidently unaware of F. R. Moulton's Differential Equations [Macmillan, New York, 1930], in Chapter III of which this question is thoroughly treated. *W. T. Reid.*

Hukuhara, Masuo. Théorèmes fondamentaux de la théorie des équations différentielles ordinaires. II. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. 2, 1–25 (1941). [MF 5635]

[Part I appeared in the same Mem. 1, 111–127 (1941); cf. these Rev. 2, 288.] The results of the earlier paper are

here extended to the deduction of theorems of existence, uniqueness, etc., for a system of differential equations

$$y_j' = f_j(x, y_1, \cdots, y_n), \qquad j = 1, 2, \cdots, n.$$

R. E. Langer (Madison, Wis.).

Picone, Mauro. Teoremi di confronto per i sistemi di equazioni differenziali ordinarie e loro conseguenze. Ann. Mat. Pura Appl. (4) **20**, 67–103 (1941). [MF 5757]

Use the notation $f(x, y_1, \cdots, y_n) = f(x, y)$ and let $[y_i(x)]$, $[z_i(x)]$ be, respectively, solutions of the two systems of ordinary differential equations

$$y_i' = f_i(x, y), \quad z_i' = g_i(x, y), \qquad i = 1, \cdots, n,$$

where the f_i are non-decreasing functions of the arguments y_i and $f_i(x, y) \leqq g_i(x, y)$. Under certain continuity conditions on the f_i and g_i, the author shows that the initial conditions $y_i(a) \leqq z_i(a)$ imply that $y_i(x) \leqq z_i(x)$, $a \leqq x$. Making use of the above theorem the paper gives sufficient conditions for solving boundary value problems of the following type:

$$y_i' = \phi_i(x) + \sum_{h=1}^{n} \phi_{ih}(x) y_h, \quad \sum_{h=1}^{n} (a_{ih} y_h(a) + b_{ih} y_h(b)) = \alpha_i, \qquad a \leqq x \leqq b,$$

where α_i, a_{ih}, b_{ih} are given constants. Finally, the author considers a system $y_i' = f_i(x, y, \lambda)$, $y_i(a) = \alpha_i(\lambda)$ containing the parameter λ, in which the α_i and f_i are non-decreasing functions of $\lambda, y_1, \cdots, y_n$. The theorem here gives the continuity conditions on the f_i and α_i that insure the existence of a solution $[y_i(x, \lambda)]$, which is continuous in λ for $\lambda_1 \leqq \lambda \leqq \lambda_2$, given the existence of the solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

F. G. Dressel (Durham, N. C.).

Kosik, F. Über die Konstruktion der Greenschen Funktion für Operatoren höherer Ordnungen. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **17**, 167–173 (1940). (Russian) [MF 4757]

The author gives a construction of the Green's function of an nth order linear operator $L(u)$ with n end-point boundary conditions $R_j(u) = 0$, $j = 1, 2, \cdots, n$. Neither the method used by the author nor the form obtained for Green's function are new. *J. D. Tamarkin.*

Titchmarsh, E. C. On expansions in eigenfunctions. IV. Quart. J. Math., Oxford Ser. **12**, 33–50 (1941). [MF 4670]

[The previous communications appeared in J. London Math. Soc. **14**, 274–278 (1939); Quart. J. Math., Oxford Ser. **11**, 129–145 (1940); cf. these Rev. **1**, 56; **2**, 53.] The author considers boundary value problems corresponding to differential equations of second order over an infinite interval. The solution of such problems was given by H. Weyl [Math. Ann. **68**, 220–269 (1910); Nachr. Ges. Wiss. Göttingen **1910**, 442–467] by deducing it from the theory of corresponding singular integral equations, but the purpose of the author is to obtain main theorems of Weyl independently of the theory of integral equations. Let $L = d^2/dx^2 - q(x)$, $q(x)$ continuous for $0 \leqq x < \infty$. Let $w = u + iv$, $v \neq 0$, and $f(x) = f(x, w)$ be the solution of (*) $(L - w)f = 0$, $f(0) = \sin h$, $f'(0) = -\cos h$. It is proved that there always exists a solution g of (*) such that $g(x, w) \epsilon L_2(0, \infty)$; this solution is unique in the "limit-point" case or is any solution of (*) in the "limit-circle" case. Let $G(x, y) \equiv G(x, y, w) = g(x)f(y)$ for $y < x$ and $f(x)g(y)$ for $y > x$. Let

$$\Psi_{\pm}(x, w) = -i(2\pi)^{\frac{1}{2}} \int_0^{\infty} G(x, y)\psi(y)dy,$$

$\psi(y) \epsilon L_2(0, \infty)$, $\pm$ according as $v > 0$ or $v < 0$. It is assumed that $\Psi_{\pm}$ are analytic continuations of each other, admitting only simple poles $w_1, w_2, \cdots$ on the real axis. The functions

$$\psi_n(x) = f(x, w_n)\left[\int_0^{\infty} f(y, w_n)^2 dy\right]^{-\frac{1}{2}}$$

constitute an orthonormal set over $(0, \infty)$, independent of $\psi(x)$. This set is also complete, and, in case ψ and $L\psi \epsilon L_2(0, \infty)$ and $\psi(0) \cos h + \psi'(0) \sin h = 0$, and the Wronskian of g and $\psi \to 0$ as $x \to \infty$, then $\psi(x)$ can be expanded in a "Fourier series" in $\{\psi_n\}$, which converges absolutely and uniformly over any finite interval. The conditions concerning $\Psi_{\pm}$ are certainly satisfied when, for some complex value w',

$$\int_0^{\infty}\int |G(x, y, w')|^2 dxdy < \infty.$$

J. D. Tamarkin (Providence, R. I.).

Titchmarsh, E. C. On expansions in eigenfunctions. V. Quart. J. Math., Oxford Ser. **12**, 89–107 (1941). [MF 5221]

The present paper is a continuation of the preceding paper [see the preceding review] and deals with the case where the expansion appears in the form of a Fourier-Stieltjes integral. In addition to theorems of Weyl the following, more general theorem is proved. Let $\psi(x) \epsilon L_2(0, \infty)$. Let

$$\phi(x, \lambda) = \pi^{-1}\left\{\xi(\lambda) f(x, \lambda) - \int_0^{\lambda} \xi(u) f_u(x, u)du\right\},$$

where

$$\xi(u) = \int_0^{\infty} \psi(y)\chi(y, u)dy, \quad \chi(y, \lambda) = \int_0^{\lambda} f(y, u)dk(u),$$

$k(u)$ a suitable non-decreasing function of u. Then $\phi(x, \lambda) - \phi(x, -\lambda)$ converges to $\psi(x)$ in the mean of order 2 over $(0, \infty)$. *J. D. Tamarkin* (Providence, R. I.).

Voïlokoff, N. T. Sur les invariants intégraux mixtes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 110–113 (1941). [MF 5853]

The author introduces a new kind of integral invariant of the differential system

$$\text{(I)} \qquad dx_i/dt = X_i(x_1, \cdots, x_n), \qquad i = 1, 2, \cdots, n.$$

Let C_0 denote an arbitrary p-dimensional variety in the space of the initial values $x_1^0, \cdots, x_n^0$, and let C denote the variety into which C_0 is transformed, for a given t, by the trajectories of (I). Let D represent the $(p+1)$-dimensional variety formed by the segments of the trajectories of (I) joining C_0 to C. Then, if ω_p and ω_{p+1} are differential forms of order p and $p+1$, respectively, the integral form $\int_C \omega_p + \int_D \omega_{p+1}$ is called an absolute mixed integral invariant of order p when its value does not depend on t. If C_0 is required to be closed the invariant is said to be relative. The paper deals with criteria for mixed invariants, and with ways in which they lead to knowledge of integrals of the system (I). Most of the results are extensions of earlier work of Poincaré and Goursat. *A. E. Taylor.*

Eger, Max. Sur la jacobienne d'un système de Pfaff. C. R. Acad. Sci. Paris **211**, 156–158 (1940). [MF 5341]

In his papers [C. R. Acad. Sci. Paris **204**, 92–94, 217–219 (1937); **209**, 82–84 (1939); cf. these Rev. **1**, 55] the author

has defined the Jacobian $\{\omega_0\omega_1 \cdots \omega_\lambda\}$ of a Pfaffian system and, moreover, has shown that, in the case where the ω_i are forms of the first kind attached to the manifold V_n, this Jacobian is a "canonical" manifold of λ dimensions in V_n, denoted by K_λ. In this paper is shown how the Jacobians J of the so-called "formes permises" can be expressed in the K_λ, and can be used to define canonical manifolds without the use of integrals of the first kind. This is done by means of the formula

$$J=K_\lambda(1+\Omega_0)(1+\Omega_1)\cdots(1+\Omega_\lambda),$$

where the Ω_i are the polar manifolds of ω_i, every one counted with an appropriate multiplicity. Applications are made, for example, to the case where $p+1$ of the ω_i are differentials of rational functions and the other ω_i are of the first kind.

D. J. Struik (Cambridge, Mass.).

Cartan, Élie. Sur un théorème de J. A. Schouten et W. van der Kulk. C. R. Acad. Sci. Paris **211**, 21–24 (1940). [MF 5331]

Schouten and van der Kulk [Nederl. Akad. Wetensch., Proc. **43**, 955–963, 1160–1170 (1940); these Rev. **2**, 200] have shown that it is possible, for a given system of q linear total differential equations of class $q+2p$ and $q+2p+1$, to choose the q equations in such a way that each is of class $2p+1$. A proof of this theorem is given based on Cartan's theory of systems in involution. The theorem, in a slightly generalized form, now reads as follows. Let a system of q linearly independent equations of Pfaff $\theta_i=0$, $i=1, 2, \cdots, q$, be given. Let p be the smallest integer such that

$$(\lambda_1\theta_1'+\lambda_2\theta_2'+\cdots+\lambda_q\theta_q')^{p+1}=0 \pmod{\theta_1, \theta_2, \cdots, \theta_q},$$

the λ being arbitrary parameters and the θ' exterior derivatives of the θ. In this case we can choose the q equations in such a way that each is of class $2p+1$. This theorem, moreover, cannot be improved. *D. J. Struik.*

Perron, Oskar. Das Verschwinden der Klammersymbole in der Theorie der linearen partiellen Differentialgleichungssysteme. Math. Ann. **117**, 687–693 (1941). [MF 5652]

E. Schmidt [Monatsh. Math. Phys. **48**, 426–432 (1939); cf. these Rev. **1**, 76] has proved that for a system of linear partial differential equations the brackets of Jacobi vanish even when the existence of second derivatives is not presupposed. This paper presents a new proof, simpler than that of Schmidt, and based on a method suggested by W. L. Chow [Math. Ann. **117**, 98–105 (1939); cf. these Rev. **1**, 313]. Its starting point is the existence theorem for a system of differential equations

$$\frac{dz^\nu}{dt}=P^\nu(z^1, z^2, \cdots, z^n), \qquad \nu=1, \cdots, n.$$

It is interesting to compare this proof with the proof of Schmidt's theorem given by H. Auerbach [Rec. Math. [Mat. Sbornik] N.S. **9** (**51**), 731–734 (1941); cf. these Rev. **3**, 43]. *D. J. Struik* (Cambridge, Mass.).

Melzer, L. Sur la position correcte du problème de Goursat. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 693–696 (1941). [MF 4473]

This paper is concerned with a Cauchy type problem for a system of equations of the form

$$f_{(i)}^k(u_i)_x+g_{(i)}^k(u_i)_y=\Phi_{(i)}(x, y, u_1, \cdots, u_n),$$

especially the question as to whether the solution depends continuously on the initial data. Although the concept is not introduced, the idea of characteristic can be conveniently used in stating the results. Generally speaking the author finds that the solution depends continuously on the initial data only if all the characteristics are real. The Cauchy data considered in this paper consist of certain unknowns prescribed along the x-axis and the remainder along the y-axis. The axes may or may not be characteristic.

E. W. Titt (Austin, Tex.).

Picone, Mauro. Applicazione della trasformata $r^{p/2}$ di Laplace ai problemi d'integrazione delle equazioni lineari a derivate parziali con coefficienti costanti. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 221–229. [MF 5758]

The results are the same as those in another paper [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 413–426 (1940); cf. these Rev. **3**, 44]. *D. G. Bourgin.*

Picone, Mauro. Problemi riducibili d'integrazione delle equazioni lineari a derivate parziali. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 230–238. [MF 5759]

This note states part of the results of the author's paper published in the Atti. Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 642–655 [cf. these Rev. **3**, 44].

F. G. Dressel (Durham, N. C.).

Satô, Tokui. Sur l'équation aux dérivées partielles hyperbolique $s=f(x, y, z, p, q)$. I. Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **2**, 107–123 (1941). [MF 5640]

The author applies methods used by Hukuhara [Mem. Fac. Sci. Kyûsyû Imp. Univ. A. **1**, 111–127 (1941); these Rev. **2**, 288] to the partial differential equation

(A) $$\partial^2z/\partial x\partial y=f(x, y, z, \partial z/\partial x, \partial z/\partial y).$$

The existence and uniqueness in a domain D of a solution of (A) satisfying characteristic boundary conditions $z(x, y_0)=g(x)$, $z(x_0, y)=h(y)$ is established under certain assumptions; the essential assumption is the existence of lower and upper functions $\underline{\omega}$ and $\bar{\omega}$ in D with $\partial^2\underline{\omega}(x, y)/\partial x\partial y \leqq f(x, y, z, p, q) \leqq \partial^2\bar{\omega}(x, y)/\partial x\partial y$ for $\underline{\omega}\leqq z\leqq\bar{\omega}$, $\partial\underline{\omega}/\partial x\leqq p \leqq\partial\bar{\omega}/\partial x$, $\partial\underline{\omega}/\partial y\leqq q\leqq\partial\bar{\omega}/\partial y$. The proof converts the boundary value problem into an integral equation, to which a fixed point theorem of Schauder is applied. *F. John.*

Manià, Basilio. Sopra una questione di compatibilità nel metodo variazionale. Ann. Scuola Norm. Super. Pisa (2) **9**, 79–95 (1940). [MF 5455]

This posthumous paper of Manià establishes the compatibility of the boundary value problems involving ordinary differential equations and two-point boundary conditions that arise in the application of the variational method of Picone to the solution of a certain boundary value problem for the partial differential equation $\partial^2u/\partial t^2+q(P, t)\Delta_P u = F(P, t)$. Preliminary to the proof of this result, the author first establishes by the direct methods of the calculus of variations the existence of an absolute minimum for certain types of simple integral problems. *W. T. Reid.*

Hostinsky, B. Sur la densité d'énergie d'une corde vibrante. Mathematica, Cluj **17**, 30–33 (1941). [MF 5694]

For a vibrating string fixed at the ends the author shows by straightforward computation, using the Fourier series solution, that the mean energy density $t^{-1}\int_0^t(T+V)dt$ is independent of position on the string. An alternative derivation of this result is also given. *D. G. Bourgin.*

Seth, B. R. Transverse vibrations of triangular membranes. Proc. Indian Acad. Sci., Sect. A. 12, 487–490 (1940). [MF 5540]

The author determines the spatial form of the solution of the equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

for $z=0$ on the following boundaries: (a) right isosceles triangle, (b) equilateral triangle, (c) isosceles triangle containing an angle of 120°, (d) right triangle containing an angle of 60°. *A. E. Heins* (Lafayette, Ind.).

Fantappiè, L. Il punto di vista reale e quello analitico nella teoria delle equazioni a derivate parziali. Boll. Un. Mat. Ital. (2) 3, 188–195 (1941). [MF 5584]

Let Γ, with points distinguished by t, be an analytic homeomorph of the open unit interval. If $U(t)$ and $U_N(t)$ stand for $u(x, y)|_\Gamma$ and $\partial u/\partial n|_\Gamma$, respectively, then in analogy with the case of the hyperbolic string equation we may solve the Cauchy problem for the Laplace equation in the form

$$(1)\quad u(x, y)=\tfrac{1}{2}[U(t_1)+U(t_2)]+(i/2)\int_{t_1}^{t_2}U_N(ds/dt)dt.$$

The solution is valid in a neighborhood of Γ. Suppose now Γ is a simple analytic closed curve. Then (1) passes over into the solution of the Dirichlet or Neumann problem provided it is required that u be harmonic throughout the interior of the region spanned by Γ. That is to say, the two arbitrary functions U and U_N are now determined by a single function. The Poisson integral is derived in this way. The basic ideas in this paper are essentially known [cf. J. Hadamard, Lectures on the Cauchy Problem, English edition, Yale University Press, New Haven, Conn., 1923]. *D. G. Bourgin* (Urbana, Ill.).

Bernstein, S. N. and Petrowsky, I. G. On the first boundary value problem (Dirichlet's problem) for equations of the elliptic type and on the properties of functions satisfying such equations. Uspekhi Matem. Nauk 8, 8–31 (1941). (Russian) [MF 5267]

This paper contains an expository review of the problems mentioned in the title. It is accompanied by a bibliography, by three original contributions by Keldych and Petrowsky [cf. the next three reviews] and by Russian translations of classical papers on the subject.

Petrowsky, I. G. Perron's method for the solution of Dirichlet's problem. Uspekhi Matem. Nauk 8, 107–114 (1941). (Russian) [MF 5268]

This paper contains a review of Perron's paper "Neue Behandlung der ersten Randwertaufgabe für $\Delta u=0$," Math. Z. 18, 42–54 (1923).

Petrowsky, I. G. New proof of the existence of a solution of Dirichlet's problem by the method of finite differences. Uspekhi Matem. Nauk 8, 161–170 (1941). (Russian) [MF 5269]

The author gives a remarkably simple new treatment of the Dirichlet problem for $\Delta u=0$ by means of difference equations. His method is based on a consistent use of super- and subharmonic lattice functions. It works for any number of dimensions. To prove that the solution assumes the prescribed boundary value $f(Q)$ at the particular point Q it suffices to assume that there exists a sphere Σ centered outside the domain G and having only the point Q in common with its boundary Γ ($f(Q)$ being continuous). Even this hypothesis can be weakened in various ways.

The proof consists of two steps. Consider first two plane domains $G'\subset G$ bounded by the Jordan curves Γ' and Γ, respectively, and let $\delta>0$ be the distance between Γ and Γ'. Consider an arbitrary lattice L in G with a spacing $h<\delta/3$. Let $u(P)$ be a harmonic function on L with $|u(P)|<M$. The main lemma (insuring an equicontinuity property) states that there exists a constant M' independent of L and $u(P)$, such that $|u_x(P)|<M'$ for all $P\in G'$. Since $u(P)$ is itself harmonic, this lemma implies similar inequalities for all difference-ratios of $u(P)$. It obviously suffices to prove the lemma for the case where G is the square $|x|, |y|<a$ and G' the square $|x|, |y|<a-\delta$. Consider, then, the lattice function $z=(x^2-a^2)^2(y^2-b^2)^2u_x^2(P)+C\{u^2(P)+u^2(P_1)+\cdots+u^2(P_4)\}$, where $P_1, \cdots, P_4$ are the four lattice points adjacent to P and C a constant. It is shown that, for C sufficiently large, $\Delta z\geqq 0$ so that z assumes its minimum on a boundary point. It follows that $u_x^2(P)<5M^2C/\delta^3$ for $P\in G'$.

Consider next in the familiar manner a sequence of lattices L_n in G with spacings $h_n\to 0$, and on L_n a harmonic lattice function $u^{(n)}(P)$ with boundary values tending uniformly to $f(Q)$. By the lemma it is possible to select a subsequence $u^{(n_k)}(P)$ which together with all its difference ratios converges to a function $u(P)$, harmonic in G. To see that $\lim_{P\to Q}u(P)=f(Q)$, let Σ be the circle whose existence has been postulated; the center A of Σ lies outside of G while Σ and Γ have Q, and only Q, in common. Let ρ be the radius of Σ and r the distance from A to $P\in G$. It is readily seen that for any fixed $\epsilon>0$ and for sufficiently large C the lattice-function $\bar{u}(P)=f(Q)+\epsilon+C(1/\rho-1/r)$ is subharmonic on any lattice L_n. Hence the minimum of $\bar{u}(P)-u^{(n)}(P)$ is assumed at a boundary point. Now for C sufficiently large we certainly have $f(Q)+\epsilon+C(1/\rho-1/r)>f(R)$ for any $R\in\Gamma$. Hence for n_k sufficiently large $u^{(n_k)}(P)<\bar{u}(P)$ for all lattice points of L_{n_k}, and hence $u(P)\leqq\bar{u}(P)$ for $P\in G$. Therefore $\lim_{P\to Q}u(P)\leqq\bar{u}(Q)=f(Q)+\epsilon$. A similar argument shows that $\lim_{P\to Q}u(P)\geqq f(Q)-\epsilon$. *W. Feller.*

Keldych, M. V. On the solubility and the stability of Dirichlet's problem. Uspekhi Matem. Nauk 8, 171–231 (1941). (Russian) [MF 5270]

The paper gives a very clear and concise exposition of various recent results concerning the solvability of Dirichlet's problem and also the stability of the solution when the boundary of the domain varies. In Chapter I the author gives an exposition of the generalized solution of Dirichlet's problem in the sense of Wiener, and discusses the notions of regular and irregular points. In Chapter II he discusses the notion of capacity. Chapter III is devoted to Wiener's criterion for regularity of a point, and to discussion of the behavior of the solution at an irregular point. Chapter IV contains an exposition of the harmonic measure and integral representation of the generalized solution. These chapters contain some new results and many proofs of known results are presented in a new form. The last chapter V appears to contain mostly new results. Let D be a domain not having "inner" boundary points. Let $\{D_n\}$ be a sequence of Dirichlet domains containing $\bar{D}$ and converging to D. Let $f(P)$ be continuous on the boundary Γ of D, and let $\phi(P)$ be continuous in the whole space and equal to $f(P)$ on Γ. Let $U_{n,\phi}(P)$ be the function harmonic in D_n and equal to $f(P)$ on the boundary of D_n. The sequence $\{U_{n,\phi}\}$ converges to a limit function U_f in $\bar{D}$, and uniformly in the interior of D. The Dirichlet problem is said to be stable in the

interior of D if $U_f(P)$ coincides with $u_f(P)$, the generalized solution in the sense of Wiener. A point Q of the boundary Γ is said to be a point of stability if, for each continuous $f(P)$, $U_f(Q)=\lim_{n\to\infty} U_{n,f}(Q)=f(Q)$. It is proved that a point of stability is a regular point. Among other results the author proves that a point will be a point of stability or not according as the series $\sum 2^n\lambda_n$ diverges or converges; here λ_n is the capacity of the open set O_n of points of $C\bar{D}$ whose distances from Q are between 2^{-n+1} and 2^{-n}. If the sequence $\{U_{n,f}\}$ converges to $f(P)$ uniformly on Γ, no matter what is the continuous function f, the Dirichlet problem is called stable in $\bar{D}$. It is proved that the Dirichlet problem is stable in $\bar{D}$ if each point of Γ is a point of stability. It is also proved that if the Dirichlet problem is stable in the interior of D and is solvable, then it is stable in $\bar{D}$. In order that every solvable Dirichlet problem be stable in $\bar{D}$ it is necessary and sufficient that the set of points of non-stability coincide with the set of irregular points. Among various "Gegenbeispiele" we mention an example showing existence of a simply connected D bounded by a simple Jordan surface S of finite area and such that the Dirichlet problem is solvable for any continuous data on S, and at the same time is not stable in the interior of D, while the set of instability has superficial measure zero.

J. D. Tamarkin (Providence, R. I.).

Rosenblatt, Alfred. On Green's function of bounded domains in Euclidean space of three dimensions. Revista Ci., Lima **43**, 291–318 (1941). (Spanish) [MF 5737]

An amplification of the proofs of the results given in Actas Acad. Ci. Lima **4**, 42–52 (1941); cf. these Rev. **3**, 46.

E. F. Beckenbach (Ann Arbor, Mich.).

Kupradze, V. D. On the solution of the Dirichlet problem for multiply-connected domains. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 569–571 (1940). (Russian) [MF 5283]

The author discusses the Dirichlet problem for a space domain T bounded by smooth surfaces $S=\sum_{k=0}^{n} S_k$ and for a continuous boundary value function $f(P)$; T might or might not be bounded. A solution is sought in the form of the sum of a potential of a double layer and a potential of a single layer (mixed potential $u(p)$):

$$u(p)=\frac{1}{2\pi}\int\int_S \mu(Q)\left[\frac{\cos\varphi}{r^2(pQ)}+\frac{C}{r(pQ)}\right]dS_Q,$$

where C is a negative constant [see E. Goursat, Cours d'Analyse, 1927, vol. 3, p. 515]. The solution reduces to the solution of the integral equation

$$-\mu(P)+\lambda\int\int_S \mu(Q)K(P,Q)dS_Q=-f(P),$$

where $\lambda=+1$ and

$$K=\frac{\cos\varphi}{r^2(PQ)}+\frac{C}{r(PQ)}.$$

An ingenious proof is given that $\lambda=+1$ is not a characteristic value for the corresponding homogeneous equation, so that the solution is unique for $\lambda=+1$. See N. I. Mushelišvili [Mitt. Georg. Abt. Akad. Wiss. USSR **1**, 96–106, 169–170 (1940); cf. these Rev. **1**, 314] for a different reduction of the problem to the solution of an integral equation and proof that characteristic values are not involved.

E. F. Beckenbach (Ann Arbor, Mich.).

Wolf, František. The Poisson integral. A study in the uniqueness of harmonic functions. Acta Math. **74**, 65–100 (1941). [MF 5155]

It is known that a harmonic function may have zero boundary values on the circumference of a circle (for approach along radii, etc.) and still not vanish within the circle. Aside from the more or less tautologous condition of the uniform absolute continuity of $\int u(r,\theta)d\theta$, the familiar conditions that insure uniqueness are of the over-all type $|u(r,\theta)|<\psi(\theta)$, where $\psi(\theta)$ is summable L. The present paper is a study of the restrictions on the order of growth of a harmonic function, as a function of r, which are sufficient to insure uniqueness, and, in particular, uniqueness of representation in terms of the Poisson integral. By means of conformal representation, simply connected domains D with rectifiable boundaries are reduced to the circle.

The paper consists of a collection of detailed theorems, with complex hypotheses, which contain simple special cases of considerable generality. Thus, if (i) $u(r,\theta)$ is harmonic in the unit circle C, (ii) converges to zero at every boundary point θ_0 as $(r,\theta)\to(1,\theta_0)$ in any sector with vertex θ_0 (roughly, for paths not tangent to the circumference), (iii) for every $\epsilon>0$ there is an $R<1$ such that $|u(r,\theta)|\leqq\exp[\epsilon(1-r)^{-m}]$ for $r>R$, then $u\equiv 0$. If $m<1$, the sector about the radius, with vertex θ_0, in which convergence to zero is required, may be arbitrarily small. If it is desired to restrict oneself to convergence to zero only along the radii themselves, an order of growth along radii $u(r,\theta)=o[(1-r)^{-m}]$ insures that there is a reducible set $\mathfrak{R}$ on the circumference such that $u(r,\theta)$ takes continuously the value zero at every boundary point not in $\mathfrak{R}$. At an isolated point of $\mathfrak{R}$ the corresponding complex function has a pole of finite order. If there is a neighborhood of a point z_0 of the boundary of D such that when this neighborhood is transformed conformally into the interior of the unit circle C the resulting function is given by a Poisson integral on C, the original function is a "restricted Poisson integral in z_0," $\mathrm{RPI}_D(z_0)$; likewise, if K is a portion of the boundary of D which is also a portion of the boundary of $D_1\subset D$, and the conformal transformation of D_1 into C yields a function given by the Poisson integral, then $u=\mathrm{RPI}_D(K)$.

The general method of the paper is to devise theorems such that $u=\mathrm{RPI}_D(z_0)$ except for certain reducible exceptional sets $\mathfrak{R}$, and to study the isolated points of these exceptional sets. The first problem is solved by studying the order of growth of $|u(r,\theta)|$ in the neighborhood of certain families of curves which cut the circumference C at non-zero angles. The second problem allows the application of theorems of Lindelöf, Carleman, etc., to isolated singular points.

G. C. Evans (Berkeley, Calif.).

Muschelišvili, N. und Avazašvili, D. Über die Lösung der fundamentalen Randwertaufgaben des logarithmischen Potentials. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **7**, 1–24 (1940). (Russian. German summary) [MF 5301]

Let S be a domain bounded outside by a contour L_0 and inside by n contours $L_1, L_2, \cdots, L_n$. The authors consider the modified Dirichlet problem: to find a function harmonic in S and assuming on L_k $(k=0,1,2,\cdots,n)$ values $f+a_k$, where f is a given function and a_k constants, $a_0=0$. By using fundamental properties of the Cauchy integral they

show that this problem can be reduced to a Fredholm integral equation

$$(*) \qquad \nu(\sigma)-\int_L K(\sigma, s)\nu(s)ds=F(s),$$

while the classical theory reduces to the equation where the integral term figures with plus sign. The advantage of the equation (*) is that the corresponding homogeneous equation admits only one non-trivial solution, while the equation of the classical theory has n solutions. The authors discuss applications of their method to the Dirichlet and Neumann problems. *J. D. Tamarkin* (Providence, R. I.).

Magnaradze, L. Fundamentale Randwertaufgaben der Potentialtheorie für Flächen mit Ecklinien. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **7**, 25–46 (1940). (Russian. German summary) [MF 5302]

Fundamental problems of the theory of logarithmic potential in the case where the boundary has corners were solved by Radon [Akad. Wiss. Wien, S.-B. IIa. **128** (1919)]. In the present paper the author extends Radon's method to the solution of analogous problems of the theory of Newtonian potential, in the case where the boundary surface S may have a denumerable set of edges, but is assumed to have area in the sense of Lebesgue and also to satisfy the condition $\int_S |d\omega_P|$ is finite, where $d\omega_P$ is the element of the solid angle which is determined by the element of the surface S relative to a fixed point P. The author applies his method also to give a solution of Stokes' theory of viscous liquids and of the fundamental boundary value problem of the theory of elasticity. *J. D. Tamarkin.*

Zamoreff, A. A. Problème inverse plan de la théorie du potentiel. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 872–874 (1941). [MF 5516]

The author outlines a method of obtaining the form of a mass distribution in two dimensions when the derivatives of the potential of the mass are known on a line exterior to the mass. He assumes that the mass is of constant density and that the boundaries of the mass are analytic curves of a special type. The problem consists in finding the equations of these curves in terms of the given data, no question of existence being involved. *J. W. Green.*

Privaloff, I. I. et Sagatelian, W. W. Sur le potentiel de double couche dans l'espace. Rec. Math. [Mat. Sbornik] N.S. **9** (51), 429–436 (1941). (Russian. French summary) [MF 4565]

The authors consider potentials of double layers, of the form $\int_S (\cos\psi/\overline{QP}^2)d\mu(e)$, where S is a surface satisfying the condition $\theta<\text{const.}\ \overline{MN}^\alpha$, θ is the angle between the normals at M and N, $\alpha>0$ ("surface of Liapounoff"), and $\mu(e)=\mu_1(e)-\mu_2(e)$, where $\mu_1(e)$, $\mu_2(e)$ are non-negative, additive functions of open sets e on S. In extending the classical result for potentials of double layers, the authors prove that for almost all points Q_0 of S, if $P\to Q_0$ along the normal at Q_0,

$$\int_S \frac{\cos\psi}{\overline{QP}^2}d\mu(e)\to\int_S \frac{\cos\psi_0}{\overline{QQ_0}^2}d\mu(e)\pm 2\pi f(Q_0),$$

where $f(Q_0)$ is an integrable function determined by $f(Q_0)=\lim_{t\to 0}(\mu(\omega_t)/|\omega_t|)$, where ω_t is the segment of the surface cut by the sphere with Q_0 as center and radius t. Here the sign + or − should be taken according as $P\to Q_0$ from inside or outside of S. *J. D. Tamarkin.*

Privalov, I. Sur la définition d'une fonction subharmonique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 281–284 (1941). (Russian. French summary) [MF 5835]

Let E_1 be a bounded closed set of capacity zero, in a p-dimensional Euclidean space. Let $\overline{\Delta}^* u$ be the upper generalized Laplacian of u (in the sense of Blaschke or of Privalov). The author proves that, if u is bounded above and upper semi-continuous in a neighborhood of E_1, and if in this neighborhood $\overline{\Delta}^* u(Q)>-\infty$ everywhere and $\overline{\Delta}^* u(Q)\geqq 0$ almost everywhere, then u is subharmonic in a domain containing E_1. *J. D. Tamarkin.*

Amerio, Luigi. Sul prolungamento analitico delle funzioni armoniche. Portugaliae Math. **2**, 173–176 (1941). [MF 5692]

The author presents a proof without resorting to conformal mapping of the fact that a harmonic function assuming analytic values on a segment of an analytic curve can be analytically extended across that curve. This is done by setting up the corresponding Fredholm integral equation

$$\mu(s)=\int_B K(s, \sigma)\mu(\sigma)d\sigma+\pi^{-1}f(s)$$

for the density $\mu(s)$ of the double layer whose potential is the given harmonic function u. Since $K(s, \sigma)$ is analytic in s on an analytic piece of S, it can be expanded into a uniformly convergent power series and the integration above can be performed termwise. The result is a uniformly convergent series, and so an analytic function for $\mu(s)$. Hence u is analytic on the analytic piece of boundary and so can be extended across it. *J. W. Green* (Rochester, N. Y.).

Ridder, J. Harmonische, subharmonische und analytische Funktionen. Ann. Scuola Norm. Super. Pisa (2) **9**, 277–287 (1940). [MF 5467]

In the first part of the paper the author proves the following theorem and shows its applications to the theory of analytic, harmonic and subharmonic functions. Let C be a closed rectifiable curve, $p(x, y)$, $q(x, y)$ real valued functions continuous in the closed domain B bounded by C, and such that $\int_R p dx+q dy\leqq 0$, where the integral is taken over any rectangle with sides parallel to axes situated in the interior of B (open domain bounded by C). Then also $\int_C p dx+q dy\leqq 0$. In the second part of the paper the author discusses various conditions characterizing harmonic, subharmonic and analytic functions. Among several theorems, partly known, partly new, proved by the author we mention the following. (I) Let v be harmonic in B and greater than 0 and u have first partial derivatives continuous in B. If the interval function

$$\Psi(R)=\int_R (v\partial u/\partial n-u\partial v/\partial n)ds$$

satisfies the following conditions in B: (1) $D^+\Psi$ can be $+\infty$ at no more than a denumerable set of points; (2) almost everywhere $D^-\Psi\leqq 0$, then u is subharmonic in B. (II) The function u is harmonic in B if it has continuous first partial derivatives and, in addition, if (1) $\partial u/\partial x$, $\partial u/\partial y$ have finite extreme derivatives with respect to x and y, except perhaps at a denumerable set of points, (2) at almost all points where $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$ both exist the Laplace equation is satisfied. *J. D. Tamarkin* (Providence, R. I.).

Brelot, M. Quelques propriétés locales à la frontière des fonctions harmoniques ou sous-harmoniques. Bull. Sci. Math. (2) **64**, 153–162 (1940). [MF 5231]

This paper continues the study of harmonic and subharmonic functions, with reference to sets "effilés" at a point and the generalizations from domains to open sets of the ideas connected with sequence solutions. Let ω be a bounded open set composed of domains ω_i, F the frontier of ω, F_i the frontier of ω_i, E a subset of F. The set E is negligible with respect to ω if for every i the common part of E and F_i is of zero interior harmonic measure with respect to ω_i.

Given the function u in ω and the point P on F, $l_P u$ denotes the inferior and $L_P u$ the superior limit of u in ω at P; these functions are defined on F. If then $C_P E$, the complement of E (with respect to F), is not empty in the neighborhood of a particular point Q of F, it is of interest to consider the inferior limit $I(Q)$ at Q of the superior function $L_P u$ on $C_P E$. Evidently $l_Q u \leqq I(Q)$.

Among several theorems, A. J. Maria's well-known theorem on the upper bound or superior limit of a potential finds a generalization in the following: that if u is harmonic and bounded in ω and "majeur" in the portion γ of ω in a spherical neighborhood of Q (that is, u is identical with the Wiener function H_λ^γ [Acta Litt. Sci. Szeged **9**, 133–153 (1939); these Rev. **1**, 121], where $\lambda = L_P u$) and E is a negligible subset of F, then $I(Q) = l_Q u$. *G. C. Evans.*

Evans, G. C. Continua of minimum capacity. Bull. Amer. Math. Soc. **47**, 717–733 (1941). [MF 5476]

The principal problem undertaken and solved in this paper, which was the author's presidential address to the American Mathematical Society, is to prove the existence of a surface spanning a given closed curve s in space and yielding a minimum capacity among all such surfaces. A somewhat analogous problem was considered by Poincaré, who examined the smooth surfaces containing a given volume and concluded that among them the sphere gave the least capacity. The present paper opens with a derivation of Poincaré's result, and also of a general condition that a smooth surface S offer a minimum capacity among nearby smooth surfaces. This condition is that $dV/dn^+ + dV/dn^- = 0$, where V is the conductor potential on S and n^+ and n^- are the two normals to S.

The paper next passes to consideration of similar problems in the plane which are the natural extensions of the problems mentioned above. First there is a proof of the fact that among closed sets F of given measure the circle gives a minimum capacity. The proof follows lines laid out by Szegö for similar problems and makes use of the Schwarz and isoperimetric inequalities. The relation between problems on capacity and problems on conformal mapping is introduced and a number of results obtained. First, using the methods of conformal mapping, another proof is given of the fact that the circle gives the least capacity for sets F of given measure, this time in the special case F is a single continuum. Then, using this fact and the elementary properties of capacity, several theorems on capacity and conformal mapping fall out. Among them are an inequality of Szegö, $l \leqq 4K$, where K is the capacity and l the linear measure of the projection of F on some line, Koebe's Verzerrungssatz and results similar to Bloch's theorem. The problem in the plane most nearly analogous to the main theorem of this paper is the problem of finding a continuum containing $n \geqq 2$ fixed points and yielding a minimum capacity. When $n = 2$, the inequality of Szegö gives immediately the line segment joining the two points as solution. When $n > 2$, the problem has been solved by Grötsch.

The author then turns to the existence theorem mentioned in the first paragraph, and shows the existence of a single surface S which furnishes the minimum capacity among all surfaces S' with the following properties: (i) S' is bounded and every point of s is a limit point of S'; (ii) except near s, S' consists of a finite number of smooth surface elements; (iii) every closed curve looping s has a point in common with S'. [For a more precise statement of the conditions on S and s, and for an outline of the proofs, see the reviews of two papers by the author, Proc. Nat. Acad. Sci. U. S. A. **26**, 489–491, 664–667 (1940); these Rev. **2**, 58.] *J. W. Green* (Rochester, N. Y.).

Brodsky, G. A. Über eine Grenzwertaufgabe der Theorie der biharmonischen Funktionen. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 208–209 (1941). [MF 4830]

The author uses a relation established in a previous paper [Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. (1941)] to prove that the biharmonic function

$$F(r, \theta) = (1 - r^2) \sum_{i=1}^{\infty} 2^i r^{2^i} \cos 2^i \theta$$

is bounded in the unit circle but has no radial limit on a set of positive measure. *L. H. Loomis.*

Vecoua, I. N. and Kharasoff, D. F. Remarks on the method of Fourier. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 647–650 (1940). (Russian. Georgian summary) [MF 5285]

It is shown that all real solutions of the partial equation $\Delta^n u + a_1 \Delta^{n-1} u + \cdots + a_{n-1} \Delta u + a_n u = 0$, where u is an unknown function of two variables and Δu is the Laplacian of u, can be represented as sums, finite or denumerable, of solutions of the type $f(r)\phi(\theta)$, where r, θ are polar coordinates. *J. D. Tamarkin* (Providence, R. I.).

Tautz, G. Zur Theorie der elliptischen Differentialgleichungen. I. Math. Ann. **117**, 694–726 (1941). [MF 5653]

The author considers the differential equation

$$u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

which in integral form becomes

$$-\int_{\omega^*} \frac{\partial u}{\partial n} ds + \int\int_{\omega} \left(\frac{\partial u}{\partial x} dA + \frac{\partial u}{\partial y} dB + u dC \right) = F(\omega).$$

He assumes that the set functions $A(e)$, $B(e)$, $C(e)$ and $F(e)$ are completely additive on a bounded region G on which they satisfy a condition of the form $\|\varphi(r \cdot G)\| \leqq C r^{1+\alpha}$, $\alpha > 0$, $\|\varphi(e)\|$ denoting the variation of φ over the set e. It is proved that, if K is a circle of sufficiently small radius and $f(s)$ is a bounded measurable function on K^*, there exists a unique function $u(x, y)$ of class C' interior to K, whose first derivatives also satisfy uniform Hölder conditions on closed sets interior to K, which takes on the values $f(s)$ in the sense of Fatou. An elementary solution is constructed, and the maximum principle is proved if $C(e) \leqq 0$. In the case that a, b and c satisfy uniform Hölder conditions, the adjoint operator is defined in integral form and Green's formula is shown to hold under the assumption that the solutions of

the adjoint nonhomogeneous equation also possess solutions which are of class C' [this not being proved].

C. B. Morrey, Jr. (Berkeley, Calif.).

Vecoua, Elias. Komplexe Darstellung der Lösungen elliptischer Differentialgleichungen mit Anwendungen auf Randwertprobleme. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 7, 161–253 (1940). (Russian. German summary) [MF 5308]

In this paper, the differential equation

$$\text{(A)}\quad \text{(A}_0\text{)}\qquad \mathfrak{L}_n(u)=\Delta^n u+\sum_{k=1}^{n} L_k(\Delta^{n-k}u)=\begin{cases} f(x,y)\\ 0\end{cases}$$

is investigated, where, as usual, Δ^m is the m times repeated Laplace operator, L the general linear differential operator of the kth order in two independent variables with analytic coefficients; $f(x, y)$ is an analytic function. By a theorem of E. Picard, each regular solution of equation (A) is an analytic function. Then, if $z=x+iy$, $z'=x-iy$, (A) takes on the form

$$\text{(A')}\qquad \frac{\partial^{2n}u}{\partial z^n\partial z'^n}+\sum_{k=1}^{n} L_k\left(\frac{\partial^{2n-2k}u}{\partial z^{n-k}\partial z'^{n-k}}\right)=F(z,z').$$

It is easily shown that equation (A') is equivalent to the Volterra integral equation in the complex domain:

$$\begin{aligned} u(z,z')&+\int_{z_0}^{z} a(z,z';z_1)u(z_1,z')dz_1\\ &+\int_{z_0'}^{z'} a'(z,z';z_1')u(z,z_1')dz_1'\\ &+\int_{z_0}^{z}dz_1\int_{z_0'}^{z'} b(z,z';z_1,z_1')u(z_1,z_1')dz_1'\\ &=\Gamma_n(z,z')+F_1(z,z'), \end{aligned}\tag{1}$$

where a, a' and b are certain analytic functions which can be expressed with the help of the coefficients of equation (A), z_0 and z_0' are fixed points, $\Gamma_n(z, z')$ is a harmonic function of the nth order, that is, satisfying $\Delta^n\Gamma_n=0$, and therefore has the form

$$\Gamma_n(z,z')=\sum_{k=0}^{n-1}[z^k\psi_k(z')+z'^k\varphi_k(z)],\tag{2}$$

where $\varphi_k(z)$ and $\psi_k(z')$ are arbitrary functions; and finally

$$F_1(z,z')=\int_{z_0}^{z}dz_1\int_{z_0'}^{z'}\frac{(z-z_1)^{n-1}(z'-z_1')^{n-1}}{(n-1)!^2}F(z_1,z_1')dz_1'.\tag{3}$$

When $F=0$, it is found from the solution of (1) that

$$\begin{aligned} u(z,z')&=\Gamma_n(z,z')-\int_{z_0}^{z}A(z,z';z_1)\Gamma_n(z_1,z')dz_1\\ &-\int_{z_0'}^{z'}A'(z,z';z_1')\Gamma_n(z,z_1')dz_1'\\ &+\int_{z_0}^{z}dz_1\int_{z_0'}^{z'}B(z,z';z_1,z_1')\Gamma_n(z_1,z_1')dz_1', \end{aligned}\tag{4}$$

where A, A' and B are expressed in terms of a, a' and b. (4), with the aid of (2), furnishes all the solutions of (A_0). By substituting (2) in (4) all the solutions of (A_0) are expressed in terms of $2n$ arbitrary functions ψ_k and φ_k. By taking $z'=\bar{z}$ and $\psi_k(z')=\bar{\varphi}_k(\bar{z})$, (4) gives all the real solutions of (A_0) when the coefficients of this equation are real-valued analytic functions. Solutions of the nonhomogeneous equation (A) are obtained by substituting $\Gamma_n(z, z')+F_1(z, z')$ for $\Gamma_n(z, z')$ in (4).

The equation

$$\Delta^n u+a_1\Delta^{n-1}u+\cdots+a_n u=0\tag{5}$$

is then dealt with, where the coefficients are assumed constant and $a_n\neq 0$. Two cases are considered: (1) when $x^{2n}-a_1x^{2n-2}+\cdots+(-1)^n a_n=0$ has only simple roots, and (2) when it possesses multiple roots. For case (1) the solutions have the form

$$\begin{aligned} u(z,z')&=\Gamma_n(z,z')\\ &+\sum_{j=1}^{n} d_j\int_{z_0}^{z}dz_1\int_{z_0'}^{z'}J_0(\lambda_j((z-z_1)(z-z_1'))^{\frac{1}{2}})\Gamma_n(z_1,z_1')dz_1', \end{aligned}\tag{6}$$

where the d_j are certain fixed constants and the λ_j are the roots. For case (2) analogous solutions are obtained. When

$$\Gamma_n(z,\bar{z})=(z-z_0)^{n-1}(\bar{z}-\bar{z}_0)^{n-1}\log[(z-z_0)(\bar{z}-\bar{z}_0)],$$

(4) gives an elementary solution of (A). Using this elementary solution, the author is able to prove the above mentioned Picard theorem.

The linear combinations

$$w_N=\sum_{k=0}^{n-1}\sum_{m=0}^{N}(a_{km}u_{km}+b_{km}v_{km})\tag{7}$$

are shown to approximate an arbitrary solution of (A), where u_{km}, v_{km} are a certain sequence of particular solutions of (A). With the aid of the general solution of (A) the author is able to solve certain boundary value problems which lead to certain functional equations. By a method of N. Muschelišvili for the solution of the fundamental two-dimensional boundary value problem in elasticity theory, these functional equations are reduced to the Fredholm integral equations of the second type, and the solutions to these equations are investigated under the condition that the solutions to the boundary value problem exist. Finally, an existence theorem for the boundary value problem is proved, employing Green's function of a harmonic function of the nth order for the circle. *A. Gelbart.*

Vecoua, Elias. Allgemeine Darstellung der Lösungen elliptischer Differentialgleichungen in einem mehrfach zusammenhängenden Gebiet. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 329–335 (1940). (German. Georgian summary) [MF 5276]

To every partial differential equation $L(u)=u_{xx}+u_{yy}+au_x+bu_y+cu=0$ there exists an operator P such that every solution of $L(u)=0$ regular in a simply connected domain can be represented there in the form $u=P(\phi)=\Re[K(z,\bar{z})\phi(z)+\int_{z_0}^{z}\beta(z,\bar{z},\zeta)\phi(\zeta)d\zeta]$, $\phi(\zeta)$ being a function of one complex variable. [See the preceding review and a short report by Bergman, Mitt. Forsch.-Inst. Math. Mech. Univ. Tomsk 1, 304 (1935).] ϕ is determined by u. In the present paper the author considers the case of a multiply connected domain. He proves that ϕ has (in general) the form

$$\phi(z)=\sum g_k\lg(z-a_k)+f(z),$$

where the g_k are entire functions, and states the conditions for determining ϕ in a univalent manner. *S. Bergman.*

Vecoua, I. N. Boundary value problems of the theory of linear elliptic equations with two independent variables. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 497–500 (1940). (Russian) [MF 5281]

The author considers the first boundary value problem for the equation $L(u)=u_{xx}+u_{yy}+au_x+bu_y+cu=0$ in a multiply connected domain. He uses his previous result [cf. the paper reviewed above; see also Bergman, Rec. Math. [Mat. Sbornik] 2 (44), 1169–1198 (1937)], according to which there exists for every solution u an analytic function $f(z)$ such that $u=\Re[K_1(z,\bar{z})f(z)+K_2(z,\bar{z};t)f(t)dt]$, K_1 and K_2 depending only upon L. The author proves that the above boundary value problem may be reduced to a singular integral equation for the determination of f. This equation can be replaced by an integral equation of Fredholm's type.

A. Gelbart (Raleigh, N. C.).

Cimino, Massimo. Una soluzione in grande del problema di Cauchy per una particolare equazione in tre variabili, ottenuta con un metodo di M. Picone. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 800–809 (1941). [MF 5753]

The author considers the partial differential equation of degree $2n$, $\Delta^n u(x,y,z)=0$, where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, with Cauchy initial data on the sphere $\rho=r$, where ρ, φ, θ are spherical coordinates. The paper treats two problems: that of obtaining an explicit solution and that of obtaining an existence theorem. The first is done by assuming a solution in the form of a series of spherical harmonics with coefficients which are unknown functions of ρ. These coefficients are found to satisfy an ordinary differential equation of order $2n$ whose general solution can be written down and whose arbitrary constants are determined by the given Cauchy data. In order to obtain an existence theorem, the functions appearing in the initial data are assumed to be entire functions of x, y and z, and the series is shown to converge uniformly in any bounded region excluding the origin. From known properties of series of functions satisfying $\Delta^n u=0$, it follows that all the derived series converge uniformly and the series solution satisfies the differential equation and the initial conditions. *J. W. Green.*

Viola, Tullio. Calcolo approssimato di autovalori. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 2, 71–106 (1941). [MF 5750]

A clear exposition of some known methods for approximate evaluation of characteristic values of various problems of mathematical physics. Such are (1) the method of determining λ, satisfying $L\phi+\lambda M\phi=0$, $\|\phi\|=1$, where L, M are linear operators in a Hilbert space, by means of minimizing $\|L\phi+\lambda M\phi\|$, $\phi=\sum_1^n\gamma_k\phi_k$, (2) application of Cauchy-Lipschitz method for integration of differential equations, (3) power series expansions in the neighborhood of a singular point. The proofs are usually omitted, but various bibliographical references are given. *J. D. Tamarkin* (Providence, R. I.).

Minakshisundaram, S. On the expansion of an arbitrary function in a series of eigenfunctions of boundary value problems. Proc. Indian Acad. Sci., Sect. A. 12, 462–465 (1940). [MF 5539]

Let D be a bounded open domain, simply or multiply connected, whose boundary C is composed of a finite number of regular curves in the x, y plane, and let $\omega_1(x,y)$, $\omega_2(x,y)$, $\omega_3(x,y)$, $\cdots$ be the eigenfunctions (normal and orthogonal), with corresponding positive eigenvalues $\lambda_1<\lambda_2<\lambda_3, \cdots$, of the boundary value problem:

$$\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\lambda\omega=0, \qquad \omega(x,y)=0 \text{ on } C.$$

Let $f(x,y)$ be a bounded measurable function, defined in $D+C$, and let

$$a_n=\int\int_D f(x,y)\omega_n(x,y)dxdy.$$

Let C_δ denote the circle in D with center (x,y) and radius $\delta>0$. If

$$\lim_{\delta\to 0}\frac{1}{\pi\delta^2}\int\int_{C_\delta} f(\xi,\eta)d\xi d\eta$$

exists it is called the mean limit of $f(x,y)$ at the point (x,y). The author outlines a proof of the following theorem: At a point where the mean limit of $f(x,y)$ exists

$$\lim_{t\to+0}\sum_{n=1}^{\infty} a_n\omega_n(x,y)e^{-\lambda_n t}$$

exists, and is equal to the mean limit of $f(x,y)$.

F. W. Perkins (Hanover, N. H.).

Giunti, Vittoria. Sviluppi in serie tipo Fourier di un vettore, secondo autovettori di un certo problema, e applicazione all'integrazione dell'equazione lineare a derivate parziali del 4° ordine competente al moto delle sbarre vibranti, dotate d'inerzia rotatoria. Rend. Circ. Mat. Palermo 63, 26 pp. (1941). [MF 5751]

For the fourth order partial differential equation

$$\omega_{xxxx}-A\omega_{xx}+B\omega_{tt}-C\omega_{xxtt}=f(x,t),$$

where A, B, C are constants ($A\geqq 0$, $B>0$, $C>0$), the author solves the boundary value problem $\omega(0,t)=\omega_x(0,t)=\omega(l,t)=\omega_x(l,t)=\omega(x,0)=\omega_t(x,0)=0$, $0\leqq x\leqq l$, $0\leqq t$, and also indicates when the following boundary value problem can be solved: $\omega(0,t)=\omega_x(0,t)=\omega(l,t)=\omega_x(l,t)=\omega(x,0)=\omega(x,T)=0$, $0\leqq x\leqq l$, $0\leqq t\leqq T$. In each problem ω is obtained in the form

$$\omega(x,t)=\sum_{i=0}^{\infty} c_i(t)y_i(x),$$

where the y_i are fundamental functions of the system

$$\frac{d^4y}{dx^4}+(\lambda C-A)\frac{d^2y}{dx^2}-\lambda By=0,$$

$$y(0)=y'(0)=y(l)=y'(l)=0;\ 0\leqq x\leqq l.$$

F. G. Dressel (Durham, N. C.).

Mersman, W. A. Heat conduction in an infinite composite solid. Bull. Amer. Math. Soc. 47, 956–964 (1941). [MF 5944]

The author gives a complete treatment of the boundary value problem for the temperatures in the infinite solid composed of two semi-infinite solids $x>0$ and $x<0$, of different materials, in contact at $x=0$, when the initial temperature is an arbitrary function $f(x)$. After formally determining the temperature function $U(x,t)$ with the aid of the Laplace transformation, he establishes broad sufficient conditions on $f(x)$ under which his function $U(x,t)$ satisfies all conditions of the problem. When certain requirements of regularity are added to the statement of the boundary value problem he shows that his solution is the only possible solution. *R. V. Churchill.*